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DETERMINATION OF NONLINEAR PITCHING-MOMENT

CHARACTERISTICS OF AXIALLY SYMMETRIC

MODELS FROM FREE-FLIGHT DATA

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DETERMINATION OF NONLINEAR PITCHING-MOMENT CHARACTERISTICS OF AXIALLY SYMMETRIC MODELS FROM FREE-FLIGHT DATA

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SUMMARY

An analysis is presented for the pitching and yawing motion of a symmetrical missile with a nonlinear restoring moment described by two terms, the first proportional to the resultant angle of attack, and the second proportional to the cube of the resultant angle. The solutions to the nonlinear equations of motion for zero damping and constant roll are found in closed form in terms of the elliptic integrals of the first and third kinds.

The frequency of the resultant pitching and yawing motion is shown to be a function of the maximum and minimum resultant amplitudes as well as the proportionality constants of the cubic restoring moment. The precession in the system due to the nonlinear restoring moment is also a function of these parameters. The frequency of the motion is closely approximated as a linear function of the square of the maximum and minimum amplitudes over a large part of its range of variation.

A rapid method of estimating the cubic restoring-moment coefficients from the observed frequency and amplitudes of two independent sets of free-flight data is developed and demonstrated. The method rests on the approximate linear relationship of the frequency with the maximum and minimum amplitudes of motion.

INTRODUCTION

In free-flight range experiments, the aerodynamic forces and moments acting on a missile are measured by means of very accurate observations of its motion in flight. This experimental technique requires a knowledge of the functional dependence of these aerodynamic forces and moments on the dynamic variables of the motion in order that the solution curves to the equations of motion may be obtained. The forces and moments are calculated from the parameters of the solution curves that are fitted to the motion.

This method of determining the aerodynamic forces and moments of a missile has called for solution curves which are in closed form. The motion of a missile in a free-flight range has thus been traditionally described by linear equations. Since nonlinear terms arise from both large angles of yaw and the presence of second or higher order terms in the aerodynamic moment expansion, the free-flight range technique would appear to be restricted to missiles that have linear moment systems and that fly at small angles of yaw.

The linear solutions to the equations of motion, however, can be well fitted to many motions which possess nonlinear moment systems or fly at large angles of yaw. This seems to suggest that the parameters of these linear equations would be the average or "effective" values of the coefficients of the parent nonlinear equations. These effective values show a characteristic dependence on the angle of yaw. Thus in order to completely understand the nonlinear motion, as well as to determine the proper moment parameters by experiment, it is desirable to have an analytical relationship of the dependence of these nonlinear forces and moments on the dynamic variables of the motion.

Several authors have treated the problem of nonlinear motion. Zaroodny in reference 1 has treated the special case of circular yawing motion, and a more general motion for cubic static and Magnus moments has been considered by Leitmann in reference 2. Canning in reference 3 has demonstrated the analogy between a missile with a nonlinear yawing moment and a ball rolling in a suitably shaped bowl. The most elegant treatment of nonlinear motion has been presented by Murphy in references 4, 5, and 6. Murphy attacks the problem from the standpoint of a second-order analytic equation in a complex variable. The Kryloff-Bogoliuboff techniques (ref. 7) are applied to this second-order analytic equation, and the results are presented in the form of an "amplitude plane."

The attempts to solve the equations of motion for a missile with a nonlinear restoring moment have been limited to approximate or numerical methods. While these methods have proved valuable in the analysis of nonlinear motions, the exact variation of the parameters of motion is not known. For the case of zero damping and constant roll, however, the equations of motion for a symmetrical missile with a cubic yawing moment may be solved in closed form as a function of the elliptic integrals of the first and third kinds. The object of this report is to present this closed form solution. The linear moment also will be shown so that a comparison may be drawn with the nonlinear cases.

After the equations which relate the moment parameters to the characteristics of the motion have been derived, the application to two sets of free-flight runs from the Ames supersonic free flight wind tunnel will be demonstrated.

NOMENCLATURE

A	reference area
C_D	drag coefficient, $\dfrac{\text{drag}}{qA}$
C_N	normal-force coefficient, $\dfrac{\text{normal force}}{qA}$
C_{N_σ}	rate of change of normal-force coefficient with resultant angle, $\left(\dfrac{\partial C_N}{\partial \sigma}\right)_{\sigma \to 0}$
C_m	restoring-moment coefficient, $\dfrac{\text{restoring moment}}{qAl}$
C_{m_σ}	rate of change of restoring-moment coefficient with resultant angle, $\left(\dfrac{\partial C_m}{\partial \sigma}\right)$
$C_{m_{\dot\sigma}}$	rate of change of restoring-moment coefficient with time rate of change of resultant angle parameter, $\left[\dfrac{\partial C_m}{\partial(\dot\sigma l/V)}\right]_{\dot\sigma \to 0}$
C_{m_q}	rate of change of restoring-moment coefficient with resultant angular velocity parameter, $\left[\dfrac{\partial C_m}{\partial(\dot\Theta l/V)}\right]_{\dot\Theta \to 0}$
$C_{m_{p_\sigma}}$	Magnus moment coefficient, $\dfrac{\partial}{\partial \sigma}\left[\dfrac{\partial C_m}{\partial(pl/V)}\right]_{\sigma \to 0}$
c, c_1, c_2	constants of integration
$F(\varphi,k)$	incomplete elliptic integral of the first kind
H	$\dfrac{\rho A}{2m}\left[C_{N_\sigma} - 2C_D - \left(\dfrac{l}{kt}\right)^2 (C_{m_q} + \gamma C_{m_{\dot\sigma}})\right]$
I_x, I_y, I_z	moments of inertia about roll, pitch, and yaw axes, respectively
$K(k)$	complete elliptic integral of the first kind
k_1, k_2	moduli of the elliptic integrals

k_a	axial radius of gyration
k_t	transverse radius of gyration
l	reference length
M	$\frac{\rho A l}{2m} \gamma k_t^{-2} \frac{C_m}{\sigma}$
M_o, M_1	cubic restoring-moment coefficients, defined by equation (10)
$\bar{M}_o$	$M_o + \frac{P^2}{4}$
$\bar{M}_1$	mean value of $M_1 e^{-Hx}$ over a short trajectory
m	projectile mass
P	$\frac{p}{V} \frac{I_x}{I_y}$ (gyroscopic spin)
p	angular rolling velocity
q	dynamic pressure, $\frac{1}{2} \rho V^2$
T	$\frac{\rho A}{2m} \left[C_{N_\sigma} - C_D + \left(\frac{l}{k_a}\right)^2 \gamma C_{m_{p_\sigma}} \right]$
t_1, t_2, t_3	roots of equation (16)
u, v, w	components of linear velocity of missile along x, y, and z directions
V	magnitude of missile's velocity
x	distance along the trajectory
α	angle of attack
β	angle of sideslip
Γ	$\sigma e^{-(H/2)x}$
γ	cosine of angle between missile's axis and trajectory
ϵ	resultant angle squared, σ^2
Θ	resultant angle between projectile axis and an axis fixed in space

θ polar coordinate angle of oscillatory motion, $\tan^{-1} \frac{\alpha}{\beta}$

ξ $\beta + i\alpha$

Λ_0 Heuman's lambda function

λ wave length

η_1, η_2 arguments of Heuman's lambda function

Π elliptic integral of the third kind

ρ air density

σ resultant angle of attack, $(\alpha^2 + \beta^2)^{1/2}$

σ_0 minimum resultant angle

σ_m maximum resultant angle

τ Γ^2

φ, ψ arguments of the elliptic integrals

ω frequency, radians per unit length along trajectory

$(\ )'$ derivative of quantity with respect to x

ANALYSIS

Equations of Motion

The general equations for a symmetrical missile with zero trim angle have been derived by Murphy in reference 4. The angles of attack α and β are represented by the ratios of their respective transverse component of velocity to the resultant velocity, that is, $\alpha = w/V$ and $\beta = v/V$. These values approach the exact values for small angles. Consider now the homogeneous second-order analytic equation of a complex variable for a symmetrical missile in a nonrolling coordinate system (from ref. 6):

$$\xi'' + \left(H - \frac{\gamma'}{\gamma} - iP\right)\xi' - (M + iPT)\xi = 0 \tag{1}$$

where

$\xi \qquad \beta + i\alpha$

$H \qquad \dfrac{\rho A}{2m}\left[C_{N_\sigma} - 2C_D - \left(\dfrac{l}{k_t}\right)^2 (C_{m_q} + \gamma C_{m_{\dot\sigma}})\right]$

$M \qquad \dfrac{\rho A l}{2m}\left(\gamma k_t^{-2}\,\dfrac{C_m}{\sigma}\right)$

$T \qquad \dfrac{\rho A}{2m}\left[C_{N_\sigma} - C_D + \gamma\left(\dfrac{l}{k_a}\right)^2 C_{m_{p\sigma}}\right]$

$P \qquad \dfrac{p}{V}\,\dfrac{I_x}{I_y}$

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$k_t \qquad \sqrt{\dfrac{I_y}{m}}$, transverse radius of gyration

$k_a \qquad \sqrt{\dfrac{I_x}{m}}$, axial radius of gyration

γ cosine of angle between missile's axis and trajectory

$\xi' \qquad \dfrac{d\xi}{dx}$

Assume now, for the purposes of this paper, that the geometric nonlinearities are zero; that is, the flight path angle is small enough to neglect so that the values of γ and its derivative become $\gamma = 1$ and $\gamma' = 0$.

After the polar transformation is made,

$$\xi = \sigma e^{i\theta} = \sigma(\cos\theta + i\sin\theta)$$

where $\sigma^2 = \alpha^2 + \beta^2$, and the real and imaginary parts are separated, equation (1) is transformed into a fourth order system of equations of real variables.[1]

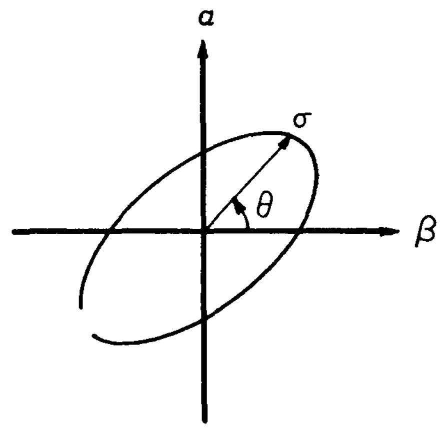
[1]Members of the Ames Research Center staff have questioned the representation of the damping moment in equations (2), principally in equation (2b). That equation (2b) may not be correct may be seen by considering purely circular motion (σ = constant, $\sigma' = 0$). In this case, the motion being steady, there should be no evidence of the unsteady derivative $C_{m_{\dot\sigma}}$, yet such terms remain. However, since the present analysis is restricted to zero damping or to small damping over a short trajectory, and since the treatment of the damping is only approximate, principal attention being given to the static moment, this question will not affect the results of this analysis in any important way.

$$\sigma'' - \sigma\theta'^2 + H\sigma' + P\sigma\theta' - M\sigma = 0 \tag{2a}$$

$$2\sigma'\theta' + \sigma\theta'' + H\sigma\theta' - P\sigma' - PT\sigma = 0 \tag{2b}$$

The variables σ and θ are shown in sketch (a); both are functions of x, the distance along the flight path.

For a completely nonlinear system, all of the coefficients H, P, M, and T would be considered functions of σ^2. This paper will consider only M to be a function of σ^2 and the remaining coefficients to be constant. Moreover, the coefficient T will be considered negligible, that is, T = 0. This implies that the roll rate, P, will not be large.



Sketch (a)

Equation (2b) may be integrated to express θ as a function of x and σ^2. Equation (2b) multiplied by σ (with T = 0) is

$$\sigma^2\theta'' + (2\sigma\sigma' + H\sigma^2)\theta' = P\sigma\sigma' \tag{3}$$

or

$$\frac{d}{dx}\left(\sigma^2 e^{Hx}\theta'\right) = \frac{Pe^{Hx}}{2}\frac{d\sigma^2}{dx} \tag{4}$$

which may be integrated once to take the form

$$\theta' = \frac{P}{2}\frac{1}{\sigma^2 e^{Hx}}\int e^{Hx} d\sigma^2 + \frac{c}{\sigma^2 e^{Hx}} \tag{5}$$

where c is a constant of integration. After the integral in equation (5) has been integrated by parts, θ' becomes

$$\theta' = \frac{P}{2} - \frac{HP}{2}\frac{1}{\sigma^2 e^{Hx}}\int \sigma^2 e^{Hx} dx + \frac{c}{\sigma^2 e^{Hx}} \tag{6}$$

By integration once more between the limits θ and θ_o, and x and x_o, where $\theta(x_o) = \theta_o$, the expression for the polar angle θ is obtained

$$\theta - \theta_o = \frac{P}{2}(x - x_o) - \frac{HP}{2}\int_{x_o}^{x}\frac{G(x)dx}{\sigma^2 e^{Hx}} + c\int_{x_o}^{x}\frac{dx}{\sigma^2 e^{Hx}} \tag{7}$$

where $G(x) = \int \sigma^2 e^{Hx} dx$.

The amount of precession in a system can be found from equation (7) as will be shown later. For this reason, equation (7) will be referred to as the precession equation in later parts of this report.

The crux of this report will lie in the solution of equation (2a). Equation (2a) may be reduced to only one dependent variable, σ, by the substitution of equation (6). Complications in the solution of equation (2a) may be reduced by removing the integral term in the equation for θ', equation (6). This may be accomplished by setting $H = 0$; that is, the damping is zero.

Yawing Motion With Constant Roll and No Damping

When the damping, H, has been set equal to zero, equation (6) may be expressed

$$\theta' = \frac{P}{2} + \frac{c}{\sigma^2} \tag{8}$$

and upon substitution into equation (2a), the equation for σ becomes

$$\sigma'' + \frac{P^2}{4}\sigma - M\sigma - \frac{c^2}{\sigma^3} = 0 \tag{9}$$

If the yawing-moment coefficient is represented by the sum of a linear and a cubic term, then M may be expressed as

$$M = -M_0 - 2M_1\sigma^2 \tag{10}$$

and equation (9) now reads

$$\sigma'' = \frac{c^2}{\sigma^3} - \left(M_0 + \frac{P^2}{4}\right)\sigma - 2M_1\sigma^3 \tag{11}$$

When σ'' has been rewritten as

$$\sigma'' = \frac{d}{dx}\frac{d\sigma}{dx} = \frac{d\sigma}{dx}\frac{d}{d\sigma}\frac{d\sigma}{dx} = \frac{1}{2}\frac{d}{d\sigma}\left(\frac{d\sigma}{dx}\right)^2$$

equation (11) assumes the following form:

$$\frac{d}{d\sigma}\left(\frac{d\sigma}{dx}\right)^2 = \frac{2c^2}{\sigma^3} - 2\overline{M}_0\sigma - 4M_1\sigma^3 \tag{12}$$

where $\overline{M}_O = M_O + P^2/4 > 0$ is always satisfied for statically stable missiles, and is the criterion for proper spin stabilization for statically unstable missiles.

Equation (12) may now be integrated once to give

$$\left(\frac{d\sigma}{dx}\right)^2 = -\frac{c^2}{\sigma^2} - \overline{M}_O\sigma^2 - M_1\sigma^4 + c_1 \tag{13}$$

where c_1 is a constant of integration. The constant c_1 may be evaluated from the boundary condition that $\sigma = \sigma_m$ is a maximum at $d\sigma/dx = 0$. Thus

$$c_1 = \frac{c^2}{\sigma_m^2} + \overline{M}_O\sigma_m^2 + M_1\sigma_m^4 \tag{14}$$

When both sides of equation (13) have been multiplied by σ^2, the equation becomes

$$\frac{1}{4}\left(\frac{d\sigma^2}{dx}\right)^2 = -c^2 + c_1\sigma^2 - \overline{M}_O\sigma^4 - M_1\sigma^6 \tag{15}$$

and upon the substitution $\epsilon = \sigma^2$, assumes the form

$$\frac{1}{4}\left(\frac{d\epsilon}{dx}\right)^2 = -c^2 + c_1\epsilon - \overline{M}_O\epsilon^2 - M_1\epsilon^3 \tag{16}$$

The square root may now be taken of both sides of equation (16) and the variables separated. After integration the solution would be in the form of an elliptic integral. Before the integral can be put in standard form, however, there are several points to clear up concerning the continuity of $d\epsilon/dx$. The quantity $(d\epsilon/dx)^2$ is defined (eq. (16)) as a cubic equation in ϵ, and, having at least one real root, will take on both positive and negative values, depending upon the value of ϵ. It is necessary, then, to define the domain of ϵ so that $(d\epsilon/dx)^2 \geq 0$, or, in other words, so that $d\epsilon/dx$ is real. This may be accomplished by examining the roots of equation (16); that is, the maximum and minimum values of $\epsilon = \sigma^2$. The roots of equation (16) are

$$\left.\begin{aligned}
t_1 &= \sigma_m^2 \\
t_2 &= \frac{1}{2}\left[-\left(\sigma_m^2 + \frac{\overline{M}_O}{M_1}\right) + \sqrt{\left(\sigma_m^2 + \frac{\overline{M}_O}{M_1}\right)^2 + \frac{4c^2}{M_1\sigma_m^2}}\right] \\
t_3 &= \frac{1}{2}\left[-\left(\sigma_m^2 + \frac{\overline{M}_O}{M_1}\right) - \sqrt{\left(\sigma_m^2 + \frac{\overline{M}_O}{M_1}\right)^2 + \frac{4c^2}{M_1\sigma_m^2}}\right]
\end{aligned}\right\} \tag{17}$$

<u>Case 1: A yawing moment that grows faster than a linear moment</u>.- For this case both $\overline{M}_o$ and M_1 are positive. From equations (17) it can be seen that $t_3 > 0$ and $t_2 > 0$. The quantity $(d\epsilon/dx)^2$ assumes positive or negative values in the following intervals and is shown in sketch (b).

$$\left(\frac{d\epsilon}{dx}\right)^2 > 0 \quad \text{for} \quad -\infty < \epsilon < t_3$$

$$\left(\frac{d\epsilon}{dx}\right)^2 < 0 \quad \text{for} \quad t_3 < \epsilon < t_2$$

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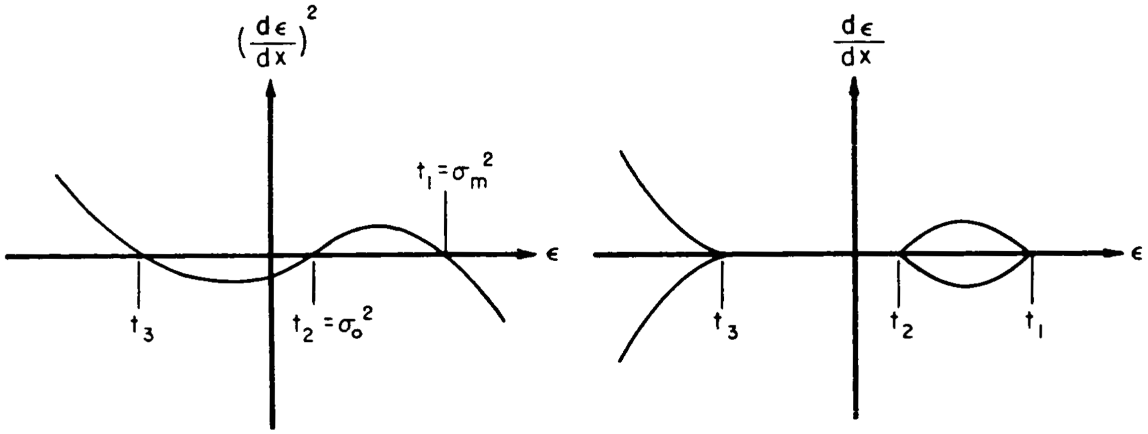
$$\left(\frac{d\epsilon}{dx}\right)^2 > 0 \quad \text{for} \quad t_2 < \epsilon < t_1$$

$$\left(\frac{d\epsilon}{dx}\right)^2 < 0 \quad \text{for} \quad t_1 < \epsilon < \infty$$



Sketch (b)

The domain of ϵ for continuous oscillatory motion will be $t_2 \leq \epsilon \leq t_1$: $\epsilon = t_1 = \sigma_m^2$ will be the maximum yaw angle squared, and $\epsilon = t_2 = \sigma_o^2$ will represent the square of the minimum yaw angle. With this in mind equation (16) may now be integrated.

Taking the positive square root of equation (16), separating the variables, and integrating between the limits ϵ and t_2, and x and x_o, where $x(t_2) = x_o$:

$$x - x_o = \frac{1}{2\sqrt{M_1}} \int_{t_2}^{\epsilon} \frac{d\epsilon_1}{\sqrt{-\frac{c^2}{M_1} + \frac{c_1}{M_1}\epsilon_1 - \frac{\overline{M}_o}{M_1}\epsilon_1^2 - \epsilon_1^3}} \tag{23a}$$

where

$$M_1 > 0$$

Equation (23a) may be written in terms of t_1, t_2, and t_3,

$$x - x_o = \frac{1}{2\sqrt{M_1}} \int_{t_2}^{\epsilon} \frac{d\epsilon_1}{\sqrt{(t_1 - \epsilon_1)(\epsilon_1 - t_2)(\epsilon_1 - t_3)}} \tag{23b}$$

where

$$t_1 \geq \epsilon \geq t_2 \geq 0 > t_3$$

The integral in equation (23b) may be transformed into an elliptic integral of the first kind by the transformation (eq. 235.00, ref. 8)

$$\sin \varphi = \sqrt{\frac{t_1 - t_3}{t_1 - t_2} \frac{\epsilon - t_2}{\epsilon - t_3}} \tag{24}$$

Equation (23b) is thus transformed into

$$x - x_o = \frac{1}{\sqrt{M_1}\sqrt{t_1 - t_3}} \int_{o}^{\varphi} \frac{d\varphi_1}{\sqrt{1 - k_1^2 \sin^2\varphi_1}} = \frac{F(\varphi,k_1)}{\sqrt{M_1}\sqrt{t_1 - t_3}} \tag{25}$$

where $F(\varphi,k_1)$ is an elliptic integral of the first kind

$$k_1^2 = \frac{t_1 - t_2}{t_1 - t_3} = \frac{\sigma_m^2 - \sigma_o^2}{\sigma_m^2 - t_3}$$

$$\sin \varphi = \sqrt{\frac{t_1 - t_3}{t_1 - t_2} \frac{\epsilon - t_2}{\epsilon - t_3}}$$

Although equation (25) is the exact solution for equation (16) for $\epsilon = \sigma^2$, it is actually in an inverse form; that is, x is written as a function of ϵ, rather than ϵ as a function of x. However, the Jacobian elliptic functions can be used to express ϵ as a function of x.

By means of the Jacobian elliptic functions described in reference 8, equation (25) may be inverted to read

$$\mathrm{am}\left[\sqrt{M_1}\sqrt{t_1 - t_3}\,(x - x_o)\right] = \varphi \tag{26}$$

$$\mathrm{sn}\left[\sqrt{M_1}\sqrt{t_1 - t_3}\,(x - x_o)\right] = \sin \varphi \tag{27}$$

where am and sn are the tabulated Jacobian elliptic functions. From equation (24), ϵ may be written as a function of φ

$$\epsilon = \frac{t_2 - t_3 k_1^2 \sin^2\varphi}{1 - k_1^2 \sin^2\varphi} \tag{28}$$

and ϵ becomes an explicit function of x by the use of the elliptic function (27).

$$\epsilon = \sigma^2 = \frac{t_2 - t_3 k_1^2 \mathrm{sn}^2\left[\sqrt{M_1}\sqrt{t_1 - t_3}(x - x_o)\right]}{1 - k_1^2 \mathrm{sn}^2\left[\sqrt{M_1}\sqrt{t_1 - t_3}(x - x_o)\right]} \tag{29}$$

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The quarter wave length of the motion can be found when $\varphi = \pi/2$. The value of $(x - x_o)$ is then equal to 1/4 the wave length, $(x - x_o) = \lambda/4$, and from equation (25)

$$\frac{\lambda}{4} = \frac{1}{\sqrt{M_1}\sqrt{t_1 - t_3}} K(k_1) \tag{30}$$

where $K(k_1)$ is a complete elliptic integral of the first kind.

The wave length, λ, is now seen to be a function of t_1, t_2, t_3, $\overline{M}_o$, and M_1 from equations (30) and (19). Equation (19) is rewritten here as

$$\frac{t_3}{t_1} = -\left(1 + \frac{\sigma_o^2}{\sigma_m^2} + \frac{\overline{M}_o}{M_1 \sigma_m^2}\right) \tag{31}$$

Equation (30) may be written

$$\sqrt{M_1}\,\frac{\lambda}{4}\,\sigma_m = \frac{K(k_1)}{\sqrt{1 - \frac{t_3}{t_1}}} = G_1\left(\frac{\sigma_o^2}{\sigma_m^2}, \frac{\overline{M}_o}{M_1 \sigma_m^2}\right) \tag{32}$$

Equation (32) is plotted in figure 1 and shows the dependency of the wave length on σ_m, σ_o, $\overline{M}_o$, and M_1.

The remaining problem in this case is to express the precession equation (7) in terms of analytical functions. For $H = 0$ equation (7) becomes

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + c\int_o^{x - x_o} \frac{d(x - x_o)}{\sigma^2} \tag{33}$$

This equation may also be written as

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + c\int_o^{\varphi} \frac{(dx/d\varphi)d\varphi_1}{\sigma^2} \tag{34}$$

From equation (25), $dx/d\varphi$ may be expressed as

$$\frac{dx}{d\varphi} = \frac{1}{\sqrt{M_1}\sqrt{t_1 - t_3}} \frac{1}{\sqrt{1 - k_1{}^2\sin^2\varphi}} \tag{35}$$

and by use of equation (28), the following integral may be evaluated:

$$I = \int_o^{\varphi} \frac{(dx/d\varphi)d\varphi_1}{\sigma^2} = \frac{1}{t_2\sqrt{M_1}\sqrt{t_1 - t_3}} \int_o^{\varphi} \frac{(1 - k_1{}^2\sin^2\varphi_1)d\varphi_1}{\left(1 - \frac{t_3}{t_2} k_1{}^2\sin^2\varphi_1\right)\sqrt{1 - k_1{}^2\sin^2\varphi_1}} \tag{36}$$

After a small amount of algebraic manipulation, this integral becomes

$$I = \frac{1}{t_3\sqrt{M_1}\sqrt{t_1 - t_3}} \left[\left(\frac{t_3}{t_2} - 1\right)\int_o^{\varphi} \frac{d\varphi_1}{\left(1 - \frac{t_3}{t_2} k_1{}^2\sin^2\varphi_1\right)\sqrt{1 - k_1{}^2\sin^2\varphi_1}} + \int_o^{\varphi} \frac{d\varphi_1}{\sqrt{1 - k_1{}^2\sin^2\varphi_1}}\right] \tag{37}$$

The integral

$$\Pi(\varphi,\mu^2,k_1) = \int_o^{\varphi} \frac{d\varphi_1}{(1 - \mu^2\sin^2\varphi_1)\sqrt{1 - k_1{}^2\sin^2\varphi_1}} \tag{38}$$

is Legendre's form of the elliptic integral of the third kind. With the use of equation (25), the precession equation (34) may be expressed

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + \frac{c}{t_3}(x - x_o) + \frac{c}{\sqrt{M_1}\sqrt{t_1 - t_3}}\left(\frac{1}{t_2} - \frac{1}{t_3}\right)\Pi\left(\varphi, \frac{k_1{}^2}{\gamma_1{}^2}, k_1\right) \tag{39}$$

where $\gamma_1{}^2 = t_2/t_3$.

The constant c^2 is defined by equation (22a) and is written here as

$$c^2 = \sigma_m^2\sigma_o^2(\sigma_m^2 + \sigma_o^2)M_1 + \overline{M}_o\sigma_m^2\sigma_o^2 \tag{40}$$

and t_3 is defined by equation (19).

The polar angle θ is thus defined as a function of the elliptic integral of the third kind and a linear function of x. The elliptic integral of the third kind also arises in the analysis of the dynamics of the top, gyroscope, and spherical pendulum, which are analogous to the motion of a missile with a nonlinear yawing moment.

<u>Case 2: A yawing moment that grows slower than a linear moment</u>.- In the case where the moment grows slower than a linear moment, the coefficients M_o and M_1 are restricted to the values $M_o > 0$ and $M_1 < 0$. Moreover, if the motion is to be oscillatory, then $d^2\sigma/dx^2 < 0$, and, from equation (11),

$$\frac{c^2}{\sigma_m^4} - \overline{M}_o - 2M_1\sigma_m^2 < 0 \tag{41}$$

or, since $M_1 < 0$ and $c^2 = \sigma_o^2\sigma_m^2[(\sigma_m^2 + \sigma_o^2)M_1 + \overline{M}_o]$

$$\frac{-\overline{M}_o}{M_1\sigma_m^2} > \frac{\dfrac{\sigma_o^2}{\sigma_m^2}\left(1 + \dfrac{\sigma_o^2}{\sigma_m^2}\right) - 2}{\dfrac{\sigma_o^2}{\sigma_m^2} - 1} \tag{42}$$

To determine the domain of ϵ or, in other words, the range of ϵ for which $d\epsilon/dx$ is a continuous function, it remains to resolve the relative magnitudes of t_1, t_2, and t_3. From equation (21) it can be seen that $t_2t_3 > 0$ since $t_1 > 0$. From the inequality (42) and equation (19), the sum of t_2 and t_3 is

$$t_2 + t_3 > t_1$$

so that both t_2 and t_3 are positive. By perusing equations (17), it can be seen that $t_2 > t_3$. Now since $t_1 = \sigma_m^2$ is the maximum yaw angle squared, then $t_3 = \sigma_o^2$ is chosen as the minimum yaw angle squared, so that for Case 2, $t_1 > t_2 > t_3 > 0$. The term $(d\epsilon/dx)^2$ assumes positive or negative values in these intervals and is shown in sketch (c).

$$\left(\frac{d\epsilon}{dx}\right)^2 < 0 \quad \text{for} \quad -\infty < \epsilon < t_3$$

$$\left(\frac{d\epsilon}{dx}\right)^2 > 0 \quad \text{for} \quad t_3 < \epsilon < t_1$$

$$\left(\frac{d\epsilon}{dx}\right)^2 < 0 \quad \text{for} \quad t_1 < \epsilon < t_2$$

$$\left(\frac{d\epsilon}{dx}\right)^2 > 0 \quad \text{for} \quad t_2 < \epsilon < \infty$$

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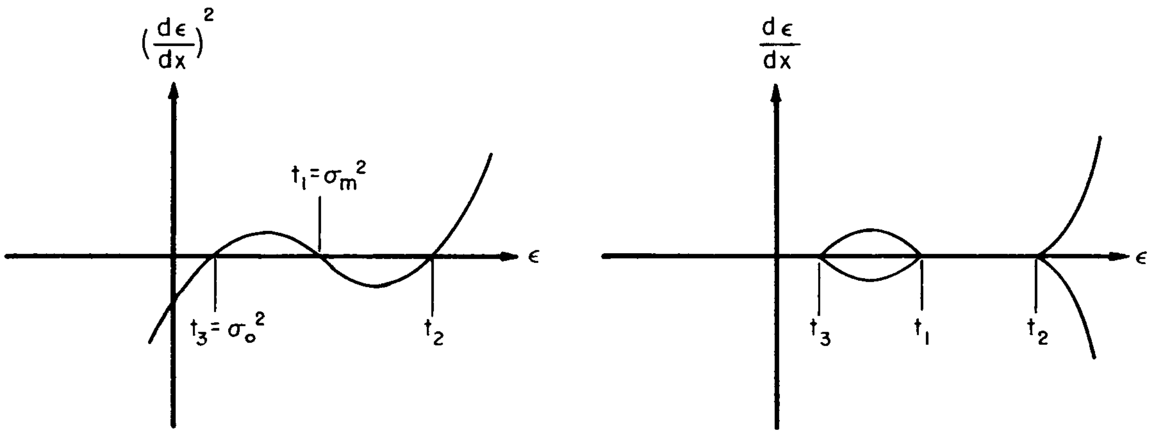

Sketch (c)

The quantity $d\epsilon/dx$ will now be continuous in the interval $t_3 \leq \epsilon \leq t_1$, and in this range the motion will be oscillatory. The motion will become less and less stable as the value of $t_1 = \sigma_m^2$ approaches the value of t_2. At the point $t_1 = t_2$ the missile will be in a state of trim. When $\epsilon > t_2$, then the motion will be unstable and the missile will overturn.

The same procedure may now be applied in integrating equation (16) as was done in Case 1. For Case 2 the limits of integration are ϵ and t_3, and x and x_o (where $x(t_3) = x_o$),

$$x - x_o = \frac{1}{2\sqrt{-M_1}} \int_{t_3}^{\epsilon} \frac{d\epsilon_1}{\sqrt{(t_1 - \epsilon_1)(t_2 - \epsilon_1)(\epsilon_1 - t_3)}} \tag{43}$$

where $-M_1 > 0$ and $t_2 > t_1 \geq \epsilon \geq t_3 > 0$.

This integral may be transformed into the form of an elliptic integral of the first kind by the transformation (ref. 8, eq. 233.00)

$$\sin \psi = \sqrt{\frac{\epsilon - t_3}{t_1 - t_3}} \tag{44}$$

and equation (43) becomes

$$x - x_o = \frac{1}{\sqrt{-M_1}\sqrt{t_2 - t_3}} F(\psi, k_2) \tag{45}$$

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where

$$k_2{}^2 = \frac{t_1 - t_3}{t_2 - t_3}$$

$$\sin \psi = \sqrt{\frac{\epsilon - t_3}{t_1 - t_3}}$$

and

$$t_1 = \sigma_m{}^2, \; t_3 = \sigma_o{}^2$$

and from equation (19)

$$t_2 = -\left(\sigma_m{}^2 + \sigma_o{}^2 + \frac{\overline{M}_o}{M_1}\right)$$

Equation (45) may be inverted to the form of the Jacobian elliptic functions so that ψ becomes a function of x.

$$\psi = \text{am}\left[\sqrt{-M_1}\sqrt{t_2 - \sigma_o{}^2}(x - x_o)\right] \tag{46}$$

$$\sin \psi = \text{sn}\left[\sqrt{-M_1}\sqrt{t_2 - \sigma_o{}^2}(x - x_o)\right] \tag{47}$$

The square of the resultant angle may be expressed as a function of ψ from equation (44)

$$\epsilon = t_3 + (t_1 - t_3)\sin^2\psi \tag{48}$$

or

$$\epsilon = \sigma_o{}^2 + (\sigma_m{}^2 - \sigma_o{}^2)\,\text{sn}^2\left[\sqrt{-M_1}\sqrt{t_2 - \sigma_o{}^2}(x - x_o)\right] \tag{49}$$

Returning to equation (45), one fourth of the wave length, $\lambda/4$, may be determined by setting $\varphi = \pi/2$:

$$\frac{\lambda}{4} = \frac{1}{\sqrt{-M_1}\sqrt{t_2 - \sigma_o^2}} K(k_2) \tag{50}$$

where $K(k_2)$ is the complete elliptic integral of the first kind. As in Case 1, λ is a function of σ_m, σ_o, and $\overline{M}_o/M_1$, and may be represented graphically when equation (50) is put into the dimensionless form

$$\sqrt{-M_1}\frac{\lambda}{4}\sigma_m = \frac{K(k_2)}{\sqrt{\dfrac{t_2}{t_1} - \dfrac{\sigma_o^2}{\sigma_m^2}}}$$

$$\sqrt{-M_1}\frac{\lambda}{4}\sigma_m = G_2\left(\frac{\sigma_o^2}{\sigma_m^2}, \frac{\overline{M}_o}{M_1\sigma_m^2}\right) \tag{51}$$

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3
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9

Equation (51) is shown in figure 2.

The equation for the polar angle θ may now be evaluated. From equation (45) $dx/d\psi$ becomes

$$\frac{dx}{d\psi} = \frac{1}{\sqrt{-M_1}\sqrt{t_2 - \sigma_o^2}} \frac{1}{\sqrt{1 - k_2^2\sin^2\psi}} \tag{52}$$

and substituting equations (52) and (48) into equation (7) for $H = 0$, the expression for θ becomes

$$\left.\begin{aligned}\theta - \theta_o &= \frac{P}{2}(x - x_o) + c\int_o^{\psi}\frac{(dx/d\psi)d\psi_1}{\sigma^2(\psi_1)} \\ \theta - \theta_o &= \frac{P}{2}(x - x_o) \\ &\quad + \frac{c}{\sigma_o^2\sqrt{-M_1}\sqrt{t_2 - \sigma_o^2}}\int_o^{\psi}\frac{d\psi_1}{\left[1 - \left(1 - \dfrac{\sigma_m^2}{\sigma_o^2}\right)\sin^2\psi_1\right]\sqrt{1 - k_2^2\sin^2\psi_1}}\end{aligned}\right\} \tag{53}$$

and noting that

$$\left(1 - \frac{\sigma_m^2}{\sigma_o^2}\right) = \left(1 - \frac{t_2}{t_3}\right) k_2^2$$

then

$$\theta - \theta_o = \frac{P}{2} (x - x_o) + \frac{c}{\sigma_o^2\sqrt{-M_1}\sqrt{t_2 - \sigma_o^{-2}}}\Pi\left(\psi, \frac{k_2^2}{\gamma_2^2}, k_2\right) \tag{54}$$

where Π is Legendre's form of the elliptic integral of the third kind and $\gamma_2^2 = t_3/t_3 - t_2$. The constant c^2 may be expressed from equation (23a) and is identical with the value of c^2 for Case 1.

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$$c^2 = \sigma_m^2\sigma_o^2(\sigma_m^2 + \sigma_o^2)M_1 + \bar{M}_o\sigma_m^2\sigma_o^2$$

<u>Case 3: An initially unstable moment which becomes stable with increasing yaw angle</u>.- In Case 3 the linear coefficient M_o is considered to be nonrestoring, $\bar{M}_o < 0$, and the nonlinear coefficient M_1 to be restoring, $M_1 > 0$. As in Case 1, it can be seen from equations (17) that $t_1 > t_2 > 0 > t_3$. The domain of ϵ for continuity will lie in the interval $t_2 \geq \epsilon \geq t_1$ and the solution of this case will have exactly the same form as Case 1. The difference will lie in the magnitude of t_3.

In Case 3, from expression (19), t_3 is bounded by

$$t_1 + t_2 > -t_3 > 0 \tag{55}$$

whereas in Case 1,

$$-t_3 > t_1 + t_2 > 0 \tag{56}$$

Thus the values of the modulus k_1^2 will be much larger for Case 3 than for Case 1. This will greatly affect the value of the elliptic integral of the first kind. The value of t_2/t_3 will also be greater for Case 3 than Case 1 and will have a large effect on the elliptic integral of the third kind in the equation for the polar angle θ.

The equations for the motion are the same as Case 1 and are listed here as

$$x - x_o = \frac{1}{\sqrt{M_1}\sqrt{\sigma_m^2 - t_3}} F(\varphi,k_1)$$

where

$$k_1^2 = \frac{\sigma_m^2 - \sigma_o^2}{\sigma_m^2 - t_3}$$

$$\sin \varphi = \sqrt{\frac{\sigma_m^2 - t_3}{\sigma_m^2 - \sigma_o^2} \frac{\epsilon - \sigma_o^2}{\epsilon - t_3}}$$

$$t_3 = -\left(\sigma_m^2 + \sigma_o^2 + \frac{\overline{M}_o}{M_1}\right)$$

and

$$\sigma_o^2 + \sigma_m^2 > -t_3 > 0$$

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The equation for the polar angle is

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + \frac{c}{t_3}(x - x_o)$$

$$+ \frac{c}{\sqrt{M_1}\sqrt{\sigma_m^2 - t_3}}\left(\frac{1}{\sigma_o^2} - \frac{1}{t_3}\right)\Pi\left(\varphi, \frac{k_1^2}{\gamma_1^2}, k_1\right)$$

where Π is the elliptic integral of the third kind and $\gamma_1^2 = \sigma_o^2/t_3$.

The motion due to a nonlinear pitching moment can be described by the use of elliptic integrals of the first and third kinds and by the use of the Jacobian elliptic functions. It would be interesting to compare these solutions with the characteristic sine solutions of a linear analysis.

Linear yawing moment.- When the yawing moment is linear, the coefficients $\overline{M}_o$ and M_1 are $\overline{M}_o > 0$ and $M_1 = 0$. Equation (16) becomes, for the linear case,

$$\frac{1}{4}\left(\frac{d\epsilon}{dx}\right)^2 = -c^2 + \left(\frac{c^2}{\sigma_m^2} + \overline{M}_o\sigma_m^2\right)\epsilon - \overline{M}_o\epsilon^2 \tag{57}$$

The zeroes of equation (57) occur at

$$\left.\begin{aligned} t_1 &= \sigma_m^2 \\ t_2 &= \frac{c^2}{\overline{M}_o\sigma_m^2} \end{aligned}\right\} \tag{58}$$

and by use of the value of c^2 from equation (40) with $M_1 = 0$, t_2 becomes $t_2 = \sigma_o^2$. Equation (57) may be directly integrated as the nonlinear cases to give

$$x - x_o = \frac{1}{2}\int_{t_2}^{\epsilon} \frac{d\epsilon_1}{\sqrt{-c^2 + \left(\frac{c^2}{\sigma_m^2} + \overline{M}_o\sigma_m^2\right)\epsilon_1 - \overline{M}_o\epsilon_1^2}} \tag{59}$$

which has the value

$$x - x_o = \frac{1}{2\omega}\left\{\sin^{-1}\left[\frac{2\omega\epsilon - \left(\frac{c^2}{\sigma_m^2} + \omega^2\sigma_m^2\right)}{\sqrt{\frac{c^2}{\sigma_m^2} + \omega^2\sigma_m^2 - 4c^2\omega^2}}\right] - \frac{\pi}{2}\right\} \tag{60}$$

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and may be simplified to the following equation:

$$\epsilon = (\sigma_m^2 - \sigma_o^2)\sin^2\omega(x - x_o) + \sigma_o^2 \tag{61}$$

where $\omega^2 = \overline{M}_o$.

The frequency for the linear case is

$$\omega = \frac{2\pi}{\lambda} \tag{62}$$

and is a constant. The expression for the polar angle θ may be determined from the precession equation (7) with $H = 0$ by substituting equation (61) into the integral.

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + c\int_{x_o}^{x} \frac{dx_1}{\sigma_o^2 + (\sigma_m^2 - \sigma_o^2)\sin^2\omega(x_1 - x_o)}$$

This integral may be integrated directly to give

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + \frac{c}{\omega\sigma_m\sigma_o}\tan^{-1}\left[\frac{\sigma_m}{\sigma_o}\tan\omega(x - x_o)\right] \tag{63}$$

and from equation (40), $c = \omega\sigma_m\sigma_o$, so that the final expression is

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + \tan^{-1}\left[\frac{\sigma_m}{\sigma_o}\tan\omega(x - x_o)\right] \tag{64}$$

The linear analysis thus gives the results in the form of the familiar circular functions, sine and arctan.

Yawing Motion With Small Damping Over a Short Trajectory

In the study of the motion of a missile in a free-flight range, the presence of damping may be an important factor in the determination of the static stability parameters. In the case of a free-flight range, the trajectory may be short, and several approximations may thus be made to adapt the damping factor to analysis for a short trajectory.[2]

Consider now equation (6) for θ':

$$\theta' = \frac{P}{2} - \frac{HP}{2}\frac{1}{\sigma^2 e^{Hx}}\int \sigma^2 e^{Hx} dx + \frac{c}{\sigma^2 e^{Hx}}$$

This expression must be substituted into equation (2a) in order to find a solution for the frequency. The main difficulty in the solution for damping in the system is due to the presence of the integral term in equation (6). The argument of this integral is always positive, and hence the magnitude of the integral will increase as the distance along the trajectory increases. The coefficient of this integral, $HP/2$, however, will in general be a small number, so that the integral term will be negligible at the beginning of the trajectory. If the damping is small, $|H| \ll 1$, and the trajectory is considered to be less than one wave length long, then the integral term in equation (6) may be neglected.

Consider now equation (6) to be

$$\theta' = \frac{P}{2} + \frac{c}{\sigma^2 e^{Hx}} \tag{65}$$

for small damping and a short trajectory, and substitute this equation into equation (2a) to get

$$\sigma'' = \frac{c^2}{\sigma^3 e^{2Hx}} - H\sigma' + \left(M - \frac{P^2}{4}\right)\sigma \tag{66}$$

When M is expressed as $M = -M_o - 2M_1\sigma^2$, equation (66) becomes

$$\sigma'' = \frac{c^2}{\sigma^3 e^{2Hx}} - H\sigma' - \left(M_o + \frac{P^2}{4}\right)\sigma - 2M_1\sigma^3 \tag{67}$$

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[2]The question of damping mentioned in connection with equations (2a) and (2b) should be kept in mind at this point. The constant damping coefficient H will represent the damping of a system when the damping is small or when the motion is nearly planar and when the value of $C_{m_{\dot{\sigma}}}$ is not a dominant part of the value of H.

It is known from linear considerations that the damping affects the amplitude in exponential form. The following transformation will thus be made

$$\sigma = \Gamma e^{-Hx/2}$$

and equation (67) may be expressed in terms of Γ:

$$\Gamma'' = \frac{c^2}{\Gamma^3} - \left(\overline{M}_0 - \frac{H^2}{4}\right)\Gamma - 2M_1 e^{-Hx}\Gamma^3 \tag{68}$$

where

$$\overline{M}_0 = M_0 + \frac{P^2}{4}$$

Equation (68) is now in the same form as equation (9) for zero damping except for the exponential factor e^{-Hx} which is a coefficient of Γ^3. This factor will have the apparent effect of changing the value of M_1 as the distance along the trajectory increases. Thus the frequency should be expected to vary along the trajectory. This should also be expected to happen since the amplitude of motion is changing.

When the following expression is used,

$$\Gamma'' = \frac{1}{2}\frac{d}{d\Gamma}\left(\frac{d\Gamma}{dx}\right)^2$$

equation (68) may be integrated once to obtain

$$\left(\frac{d\Gamma}{dx}\right)^2 = -\frac{c^2}{\Gamma^2} - \left(\overline{M}_0 - \frac{H^2}{4}\right)\Gamma^2 - 4M_1\int e^{-Hx}\Gamma^3 d\Gamma + c_2 \tag{69}$$

where c_2 is the constant of integration. The integral in equation (69) may be integrated by parts so that

$$4M_1\int e^{-Hx}\Gamma^3 d\Gamma = M_1 e^{-Hx}\Gamma^4 + HM_1\int \Gamma^4 e^{-Hx} dx \tag{70}$$

As before, for small damping ($|H| \ll 1$) and a short trajectory, the integral term on the right-hand side of equation (70) will be considered negligible compared to the integrated term which precedes it. Furthermore, the variation of the factor $M_1 e^{-Hx}$ over a short trajectory for small damping will be small, so that the factor M_1e^{-Hx} can be replaced by a mean factor $\overline{M}_1 = M_1e^{-Hx}$ which is a mean value over the trajectory. With these considerations, equation (69) may be replaced by

$$\left(\frac{d\Gamma}{dx}\right)^2 = -\frac{c^2}{\Gamma^2} - \left(\overline{M}_0 - \frac{H^2}{4}\right)\Gamma^2 - \overline{M}_1\Gamma^4 + c_2 \qquad (71)$$

This equation has the same form as the equation for constant roll and zero damping, equation (13), and it may be solved in the same manner.

One important consequence will be suffered by replacing M_1e^{-Hx} by the mean factor $\overline{M}_1$. The main effect of M_1 on the motion is on the magnitude of the wave length. When M_1e^{-Hx} is replaced by the constant $\overline{M}_1$, the wave length will be restricted to a constant mean value of the trajectory, and only the amplitude σ will vary. For small damping, however, the change in frequency of motion will be small, and the effect will be negligible over a short trajectory.

Equation (71) may now be solved by using the substitution $\Gamma^2 = \tau$ and using the results of the section on nonlinear motion for zero damping and constant roll. This may be done by replacing σ_m by Γ_m, σ_0 by Γ_0, and ϵ by τ, and using the same derived expressions for the special cases. The results will be then that the amplitude of motion will vary exponentially, and the nonlinear coefficient will assume a mean value $\overline{M}_1$.

DISCUSSION OF ANALYSIS

The relations governing a nonlinear motion have been derived for a symmetrical missile with four different cases of a cubic yawing moment. Several important characteristics of the motion can now be examined in detail.

The Frequency of Motion

The frequency of oscillation of a missile is an important parameter in the determination of the coefficients, $\overline{M}_0$ and M_1, from free-flight data. The frequency of the motion is defined as

$$\omega_e = \frac{2\pi}{\lambda} \qquad (72)$$

where λ is the wave length. The subscript e is used so that equation (72) may also be used as the effective frequency of a linear motion. Thus an effective linear value may be calculated from the equation

$$\omega^2 = \omega_e{}^2 = \frac{\rho A l}{2mk_t{}^2} C_{m_{\alpha_e}} \tag{73}$$

where $C_{m_{\alpha_e}}$ is that constant value of C_{m_α} which would give the same frequency in a linear system as would occur in the nonlinear system. Equation (72) may now be represented as a function of σ_m, σ_o, $\overline{M}_o$, and M_1 by utilizing the equations for $\lambda/4$ for the three nonlinear cases. For Case 1, equation (30) may be substituted into equation (72) to give

$$\omega_e = \frac{4\pi^2}{\lambda^2} = \frac{\pi^2}{4} M_1 \frac{t_1 - t_3}{K^2(k_1)}$$

This equation may be written in dimensionless form as

$$\frac{\omega_e{}^2}{\overline{M}_o} = \frac{\pi^2}{4} \frac{M_1 \sigma_m{}^2}{\overline{M}_o} \frac{1 - (t_3/t_1)}{K^2(k_1)}$$

When equation (19) for t_3 is used, this equation becomes

$$\frac{\omega_e{}^2}{\overline{M}_o} = \frac{\pi^2}{4} \frac{\dfrac{M_1 \sigma_m{}^2}{\overline{M}_o}\left(2 + \dfrac{\sigma_o{}^2}{\sigma_m{}^2}\right) + 1}{K^2(k_1)} \tag{74}$$

where

$$k_1{}^2 = \frac{\dfrac{M_1 \sigma_m{}^2}{\overline{M}_o}\left(1 - \dfrac{\sigma_o{}^2}{\sigma_m{}^2}\right)}{\dfrac{M_1 \sigma_m{}^2}{\overline{M}_o}\left(2 + \dfrac{\sigma_o{}^2}{\sigma_m{}^2}\right) + 1}$$

Equation (74) is plotted in figure 3. The frequency parameter, $\omega_e{}^2/M_o$, is thus seen to be a function of the dimensionless parameters $\sigma_o{}^2/\sigma_m{}^2$ and $M_1\sigma_m{}^2/\overline{M}_o$. It may be noticed from figure 3, that in spite of the nonlinear motion, these curves appear to be nearly straight lines. This would suggest that if equation (74) were expanded in a series, it could be adequately approximated by the first linear terms. Thus if the complete elliptic integral of the first kind is expanded in a series and divided term by term into equation (74), the first two terms of the resulting series give

$$\omega_e{}^2 \cong \overline{M}_o + \frac{3}{2}\left(1 + \frac{\sigma_o{}^2}{\sigma_m{}^2}\right) M_1 \sigma_m{}^2 \tag{75}$$

The details of this series expansion are shown in the appendix. It can be seen from the comparison of equation (75) with figure 3 that the approximation is very accurate for values of σ_o^2/σ_m^2 near unity. However, for σ_o^2/σ_m^2 near zero, the equation for ω_e^2 would approach the value

$$\omega_e^2 \cong \overline{M}_o + 1.44\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)M_1\sigma_m^2 \tag{76}$$

The difference between equations (76) and (75), however, is small.

An equation for the frequency for Case 2 can be derived from equation (50) in a manner analogous to that of Case 1. This equation is

$$\frac{\omega_e^2}{\overline{M}_o} = \frac{\pi^2}{4}\,\frac{\left(-\dfrac{M_1\sigma_m^2}{\overline{M}_o}\right)\left(-1-2\dfrac{\sigma_o^2}{\sigma_m^2}\right) + 1}{K^2(k_2)} \tag{77}$$

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where

$$k_2^2 = \frac{\left(-\dfrac{M_1\sigma_m^2}{\overline{M}_o}\right)\left(1 - \dfrac{\sigma_o^2}{\sigma_m^2}\right)}{\left(-\dfrac{M_1\sigma_m^2}{\overline{M}_o}\right)\left(-1-2\dfrac{\sigma_o^2}{\sigma_m^2}\right) + 1}$$

Equation (77) is plotted in figure 4. As in Cases 1 and 3, these curves are nearly linear for most of their length and deviate from linearity at the smaller values of $\omega_e^2/\overline{M}_o$. The curves in figure 4 can be approximated for the linear portion by the expression

$$\omega_e^2 = \overline{M}_o + \frac{3}{2}\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)M_1\sigma_m^2$$

This expression may be justified graphically from figure 4 and is also obtained by retaining the first two terms of the series expansion shown in the appendix.

Case 3 has essentially the same solution as Case 1, except that $\overline{M}_o < 0$. Equation (74) will still apply, but will be placed in a more convenient form when multiplied by $\overline{M}_o/M_1\sigma_m^2$. Thus, for Case 3

$$\frac{\omega_e^2}{M_1\sigma_m^2} = \frac{\pi^2}{4}\,\frac{\left(2 + \dfrac{\sigma_o^2}{\sigma_m^2}\right) + \dfrac{\overline{M}_o}{M_1\sigma_m^2}}{K^2(k_1)} \tag{78}$$

where

$$k_1^2 = \frac{1 - \frac{\sigma_o^2}{\sigma_m^2}}{\left(2 + \frac{\sigma_o^2}{\sigma_m^2}\right) + \frac{\overline{M}_o}{M_1\sigma_m^2}}$$

Equation (78) is plotted in figure 5. The frequency ω_e^2 is plotted as a function of $\overline{M}_o/M_1\sigma_m^2$ for Case 3 rather than $M_1\sigma_m^2/\overline{M}_o$ as in Case 1 so that the special case of $\overline{M}_o = 0$ could be represented for Case 3. In addition, $\overline{M}_o/M_1\sigma_m^2$ has a limited value as is shown in figure 5.

The curves for equation (78) in figure 5 are again shown to be nearly linear except at the small values of $\omega_e^2/M_1\sigma_m^2$. Equation (75) may then be used as an approximation for the linear portion of the curves. Equation (75) gives a close approximation for values of σ_o^2/σ_m^2 greater than 1/2. For values of σ_o^2/σ_m^2 near zero, the linear portion of the curve can be more accurately defined as

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$$\omega_e^2 = 1.107\overline{M}_o + 1.44\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)M_1\sigma_m^2 \tag{79}$$

Equations analogous to equation (79) could be determined for intermediate values of σ_o^2/σ_m^2 by calculating the curves from equation (78) and then approximating the slope of the linear portion of the curve by graphical means.

Expressions of the form of equation (75) will be very valuable in the evaluation of $\overline{M}_o$ and M_1 from experimental data. The application of these relations will be demonstrated in a later section.

Angular Motion in the α-β Plane and Precession

The resultant angle σ of the polar coordinates of the pitching and yawing motion, σ and θ, has been discussed, and attention will now be turned to the polar angle θ, which is defined in general by equation (7), and in particular for Cases 1-4 by equations (39), (54), and (64). In this section, the four cases for zero damping will be analyzed.

The general expression for the polar rotation is given by equation (33) and is written here as

$$\theta - \theta_o = \frac{P}{2}(x - x_o) + \sigma_m\sigma_o\sqrt{(\sigma_m^2 + \sigma_o^2)M_1 + \overline{M}_o}\int_o^{x-x_o}\frac{d(x - x_o)}{\sigma^2} \tag{80}$$

The main effect of the roll, P, may be seen from this equation. The effect of P is seen to be linear with respect to the distance along the trajectory $(x - x_o)$. The roll will thus present little difficulty in the analysis, and the main interest will be devoted to the integral term in the polar angle equation. In the remainder of this section, P will be considered zero.

The simplest case of polar motion is represented by the linear yawing moment, and the polar angle is expressed by equation (64).

$$\theta - \theta_o = \tan^{-1}\left[\frac{\sigma_m}{\sigma_o} \tan \omega(x - x_o)\right]$$

Comparison of this equation with equation (61) shows that $\theta - \theta_o$ is exactly in phase with the motion of the resultant angle σ. The amount of precession in the system is thus zero.

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In order to study the amount of precession in the three nonlinear cases, it will be convenient to study the amount of polar rotation for a quarter period of pitching and yawing motion. In this way the elliptic integrals of the third kind will be complete, and may be evaluated easily from tables.

Consider now the polar rotation for Case 1, where the yawing moment grows faster than a linear moment. This relation is given by equation (39)

$$\theta - \theta_o = \frac{c}{t_3}(x - x_o) + \frac{c}{\sqrt{M_1}\sqrt{\sigma_m^2 - t_3}}\left(\frac{1}{\sigma_o^2} - \frac{1}{t_3}\right)\Pi\left(\varphi, \frac{k_1^2}{\gamma_1^2}, k_1\right)$$

where $\Pi(\varphi, k_1^2/\gamma_1^2, k_1)$ is an elliptic integral of the third kind

$$c^2 = \sigma_m^2\sigma_o^2[(\sigma_m^2 + \sigma_o^2)M_1 + \overline{M}_o]$$

$$t_3 = -\left(\sigma_m^2 + \sigma_o^2 + \frac{\overline{M}_o}{M_1}\right)$$

$$k_1^2 = \frac{\sigma_m^2 - \sigma_o^2}{\sigma_m^2 - t_3} \text{ and } \gamma_1^2 = \frac{\sigma_o^2}{t_3}$$

For a quarter period in the yawing frequency, φ equals $\pi/2$, and the above equation may be expressed with the help of equation (30) as

$$\theta_m - \theta_o = \frac{c}{t_3\sqrt{M_1}\sqrt{\sigma_o^2 - t_3}}\left[K(k_1) + \left(\frac{t_3}{t_2} - 1\right)\Pi\left(\frac{k_1^2}{\gamma_1^2}, k_1\right)\right] \qquad (81)$$

where $\Pi(k_1^2/\gamma_1^2, k_1)$ is a complete elliptic integral of the third kind.

The complete elliptic integral of the third kind may be expressed as a function of Heuman's lambda function, $\Lambda_0(\eta,k)$, which is a tabulated function. The complete elliptic integral of the third kind may be related to the lambda function by equation 410.01 in reference 8:

$$\Pi(\alpha_1^2,k_1) = \frac{k_1^2K(k_1)}{k_1^2 - \alpha_1^2} - \frac{\pi}{2}\frac{\alpha_1^2\Lambda_0(\eta_1,k_1)}{\sqrt{\alpha_1^2(1 - \alpha_1^2)(\alpha_1^2 - k_1^2)}} \tag{82}$$

where $K(k_1)$ is a complete elliptic integral of the first kind, and

$$\sin\eta_1 = \sqrt{\frac{\alpha_1^2}{\alpha_1^2 - k_1^2}}$$

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when equation (82) is substituted into equation (81) and c and t_3 are expressed as functions of σ_m^2, σ_o^2, and $\overline{M}_o/M_1$, equation (81) reduces to the simple form

$$\theta_m - \theta_o = \frac{\pi}{2}\Lambda_0(\eta_1,k_1) \tag{83}$$

where $\Lambda_0(\eta_1,k_1)$ is Heuman's lambda function, and

$$\sin\eta_1 = \sqrt{\frac{\dfrac{M_1\sigma_m^2}{\overline{M}_o}\left(1 + \dfrac{\sigma_o^2}{\sigma_m^2}\right) + 1}{\dfrac{M_1\sigma_m^2}{\overline{M}_o}\left(1 + 2\,\dfrac{\sigma_o^2}{\sigma_m^2}\right) + 1}}$$

Equation (83) is shown graphically in figure 6. If there were no precession in the system, $\theta_m - \theta_o$ would be identically equal to 90°. The amount of precession in the system due to the nonlinearities in the yawing moment will be the difference between 90° and the actual value of $\theta_m - \theta_o$. The polar rotation for a quarter period, $(\theta_m - \theta_o)$, will always be less than 90° for Case 1. In addition, it can be seen that the amount of precession varies substantially with the value of σ_o^2/σ_m^2.

Now consider Case 2 in which the yawing moment grows slower than a linear moment. The polar rotation for Case 2 is represented by equation (54) and is expressed here for zero roll as

$$\theta - \theta_o = \frac{c}{\sigma_o^2\sqrt{-M_1}\sqrt{t_2 - \sigma_o^2}}\,\Pi\left(\psi, \frac{k_2^2}{\gamma_2^2}, k_2\right)$$

$$= \frac{\sigma_m}{\sigma_o}\sqrt{\frac{t_2}{t_2 - \sigma_o^2}}\,\Pi\left(\psi, \frac{k_2^2}{\gamma_2^2}, k_2\right) \tag{84}$$

where

$$\gamma_2{}^2 = \frac{\sigma_o{}^2}{\sigma_o{}^2 - t_2}$$

The polar rotation for a quarter period of yaw-angle frequency may be determined at $\psi = \pi/2$, so that

$$\theta_m - \theta_o = \frac{\sigma_m}{\sigma_o}\sqrt{\frac{t_2}{t_2 - \sigma_o{}^2}}\;\Pi\left(\frac{k_2{}^2}{\gamma_2{}^2}, k_2\right) \tag{85}$$

where $\Pi(k_2{}^2/\gamma_2{}^2, k_2)$ is a complete elliptic integral of the third kind.

The complete elliptic integral of the third kind may be placed in the form of Heuman's lambda function by equation 411.01 in reference 8.

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$$\Pi(\alpha_2{}^2, k_2{}^2) = \frac{K(k_2)}{1 - \alpha_2{}^2} + \frac{\pi}{2}\,\frac{\alpha_2{}^2(\Lambda_o - 1)}{\sqrt{\alpha_2{}^2(1 - \alpha_2{}^2)(\alpha_2{}^2 - k_2{}^2)}}$$

where

$$\Lambda_o = \Lambda_o(\eta_2, k_2)$$

and

$$\sin \eta_2 = \frac{1}{\sqrt{1 - \alpha_2{}^2}}$$

Upon substitution of this relationship into equation (84) and noting that $\alpha_2{}^2 = k_2{}^2/\gamma_2{}^2$ and $t_2 = -(\sigma_m{}^2 + \sigma_o{}^2 + \overline{M}_o/M_1)$, the resulting equation may be written as

$$\theta_m - \theta_o = \frac{\sigma_m}{\sigma_o}\sqrt{\frac{1 - \left(-\frac{M_1\sigma_m{}^2}{\overline{M}_o}\right)\left(1 + \frac{\sigma_o{}^2}{\sigma_m{}^2}\right)}{1 - \left(-\frac{M_1\sigma_m{}^2}{\overline{M}_o}\right)\left(1 + 2\frac{\sigma_o{}^2}{\sigma_m{}^2}\right)}}\;K(k_2) + \frac{\pi}{2}(1 - \Lambda_o) \tag{86}$$

where

$$\sin \eta_2 = \frac{\sigma_o}{\sigma_m}$$

and

$$k_2^2 = \frac{\left(1 - \frac{\sigma_o^2}{\sigma_m^2}\right)\left(- \frac{M_1 \sigma_m^2}{\overline{M}_o}\right)}{1 - \left(- \frac{M_1 \sigma_m^2}{\overline{M}_o}\right)\left(1 + 2\,\frac{\sigma_o^2}{\sigma_m^2}\right)}$$

Figure 7 illustrates equation (86).

Case 2 has two trim points for planar motion, and these points occur at $\sigma^2 = 0$ and $\sigma^2 = -\overline{M}_o/2M_1$. The second trim angle in this case, however, is unstable. Once the yaw angle has exceeded $\sigma^2 = -\overline{M}_o/2M_1$, the moment will cease to be restoring and the missile will overturn. Moreover, once σ_m^2 has exceeded the value $\sigma^2 = -\overline{M}_o/3M_1$, a precarious type of stability will exist, depending upon the value of the minimum yaw angle. This may be seen from figure 7 where the curve for circular motion, $\sigma_o^2/\sigma_m^2 = 1$, goes to infinity at $-M_1\sigma_m^2/\overline{M}_o = 1/3$.

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Figure 7 indicates that circular motion for $\sigma^2 > -\overline{M}_o/3M_1$ may not exist, but must have an elliptical nature, where the ratio of maximum and minimum yaw angles is restricted to the values shown in figure 7. The minimum yaw angle is limited to the value $\sigma_o^2 < -\overline{M}_o/3M_1$.

The equations for Case 3 (an initially unstable yawing moment which becomes more stable with increasing yaw angle) will have a form similar to that of Case 1. The difference in the equations, of course, is that the parameter $M_1\sigma_m^2/\overline{M}_o$ is negative. The value of $\theta_m - \theta_o$ for Case 3 will be expressed by equation (83).

$$\theta_m - \theta_o = \frac{\pi}{2}\,\Lambda_o(\eta_1, k_1)$$

where

$$\sin \eta_1 = \sqrt{\frac{1 + \frac{\sigma_o^2}{\sigma_m^2} - \left(- \frac{\overline{M}_o}{M_1 \sigma_m^2}\right)}{1 + \frac{2\sigma_o^2}{\sigma_m^2} - \left(- \frac{\overline{M}_o}{M_1 \sigma_m^2}\right)}}$$

and

$$k_1^2 = \frac{1 - \frac{\sigma_o^2}{\sigma_m^2}}{1 + 2\,\frac{\sigma_o^2}{\sigma_m^2} - \left(- \frac{\overline{M}_o}{M_1 \sigma_m^2}\right)}$$

The above expressions are plotted in figure 8 and show curves that are characteristically different from Case 1. The important factor in Case 3 is that a state of planar motion may exist in which the minimum yaw angle, σ_o, is different from zero. This particular motion may be studied by examining the coefficient c of the integral term in equation (33). The value of c^2 is

$$c^2 = \sigma_m^2 \sigma_o^2 [(\sigma_m^2 + \sigma_o^2)M_1 + \overline{M}_o]$$

For planar motion, c^2 is equal to 0, and the obvious case occurs at $\sigma_o = 0$. However, c may also vanish at the point where

$$-\frac{\overline{M}_o}{M_1 \sigma_m^2} = 1 + \frac{\sigma_o^2}{\sigma_m^2}$$

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At these values of $-\overline{M}_o/M_1\sigma_m^2$, $\theta_m - \theta_o$ becomes zero, as is shown in 3
figure 8. This type of planar motion may occur because there are two trim 3
points for Case 3. These points exist where $\sigma^2 = 0$ and $\sigma^2 = -\overline{M}_o/2M_1$ 9
and are the values of yaw for which the yawing moment is zero. Again, as in Case 1, $\theta_m - \theta_o$ is less than 90°.

APPLICATION OF THEORY TO DATA

Once the analytical solutions of a given type of motion have been determined, the parameters of the solution may be found for a particular missile by fitting the curves to experimental data. This is best accomplished by a curve-fitting method such as a least-squares technique. There are times, however, when a quick estimation of the stability parameters is desirable. In this section a method of rapidly estimating the parameters $\overline{M}_o$ and M_1 from a knowledge of the frequency and amplitudes of the motion will be demonstrated.

Expressions of the type shown by equations (76) and (79) have been shown to represent the frequency of motion as a linear function of $\overline{M}_o$, M_1, and σ_m^2 over nearly the entire range of frequency values. These expressions give a very convenient as well as accurate relation between the parameters $\overline{M}_o$ and M_1. The linearity and simplicity of equations (76) and (79) suggest the possibility of solving for M_1 and $\overline{M}_o$ simultaneously from two independent equations such as these. Equations (76) and (79) are stated here for the cases they represent (small values of σ_o^2/σ_m^2).

Cases 1 and 2:

$$\omega_e^2 = \overline{M}_o + 1.44\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)\sigma_m^2 M_1$$

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Case 3:

$$\omega_e^2 = 1.107\overline{M}_o + 1.44\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)\sigma_m^2 M_1$$

Similarly, equation (75) would give a better approximation for large values of σ_o^2/σ_m^2.

The two independent simultaneous equations required for the determination of $\overline{M}_o$ and M_1 may be found from the data of two independent experimental runs under like conditions. The value of $\overline{M}_o$, however, is given by the expression

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$$\overline{M}_o = M_o + \frac{P^2}{4}$$

where P is the gyroscopic spin.

In order for $\overline{M}_o$ to have the same value in each of the two simultaneous equations, $P^2/4$ must be the same in each run or else it must be negligibly small compared to M_o. A third more general alternative is that the gyroscopic spin be known for each run. For this case, a known value of $P^2/4$ is merely added to each equation and M_o replaces $\overline{M}_o$ as the second unknown.

A word should also be included on the effect of damping in the system. The analytical section on damping in this paper demonstrates that the values $\overline{M}_o$, M_1, σ_m, and σ_o should be replaced by the analogous values $\overline{M}_o - (H^2/4)$, $\overline{M}_1$, Γ_m, and Γ_o when there is damping in the system. The effect of damping on the frequency over a short trajectory will be negligible.

This method will be demonstrated by applying it to two runs, A and B, from the Ames supersonic free flight wind tunnel. The resultant amplitudes of these two runs are shown in figures 9 and 10. The free-stream Mach number was 9.6. The estimated quarter wave lengths and the maximum and minimum amplitudes are shown below.

Run A	Run B
$\sigma_m = 4.35^\circ$	$\sigma_m = 6.25^\circ$
$\sigma_o = 0^\circ$	$\sigma_o = 0.75^\circ$
$\lambda/4 = 494$ in.	$\lambda/4 = 314$ in.

The gyroscopic spin in both runs was unknown, but since the models were axially symmetric and were sabot-launched from a smooth-bore gun, it was assumed to be negligible. The damping was assumed to be negligible at the outset since the trajectories were so short. The validity of these assumptions would be tested by the quality of the fitted curves.

Since the particular type of nonlinear pitching and yawing moment was not known for this model, equation (76) was first used, and the model was found to come under Case 3; that is, the moment was initially unstable, but became more stable with increasing amplitude. This new knowledge and equation (79) were applied to the two cases, and the simultaneous equations were

$$\text{Run A} \qquad 0.00308 = 1.107\overline{M}_o + 0.00831M_1$$

$$\text{Run B} \qquad 0.00762 = 1.107\overline{M}_o + 0.01734M_1$$

from whence the values of $\overline{M}_o$ and M_1 were calculated to be

$$\overline{M}_o = -0.000472$$

$$M_1 = 0.237$$

The moment curve corresponding to these coefficients is given in figure 11, and it shows a very pronounced degree of nonlinearity. The quarter wave lengths for these values of $\overline{M}_o$ and M_1 were checked by substituting these values of $\overline{M}_o$ and M_1 into the exact equation (30). The checked values of $\lambda/4$ for runs A and B, respectively, were 41.03 feet and 26.04 feet, showing the value of the approximate equation (79).

The amplitude curves were calculated using equations (25) and (28) and are compared with the experimental data and with the amplitude curve that would be produced by a linear moment with the same frequency in figures 9 and 10. The fit shown in figure 9 for run B is shown to be excellent, and the comparison with the linear curve illustrates the possible discrepancies that can be found when a linear moment is assumed for the system. Figure 10 also illustrates this, although the fitted cubic curve does not fit the data as well in this run as it does for run B. The fitted curve for run A, however, is within the accuracy of the data. A possible reason for this discrepancy between the data and the fitted cubic curve is that the wave length is not distinctly shown in run A as it was in run B. Since run A was planar, however, a faired curve through the data could be extended through $\sigma = 0$, and $\lambda/4$ could be estimated. It was estimated for several different fairings, and the best value was demonstrated in this paper.

The quality of these two fits would lead one to believe that the damping was small. A run of a longer trajectory would have to be made to verify this. The method as it stands, however, illustrates a rapid technique for estimating the cubic moment coefficients from observations of the wave length and maximum and minimum pitching amplitude when two runs are available.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Oct. 12, 1959

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APPENDIX A

SERIES EXPANSION OF THE FREQUENCY EQUATIONS

The equation for the frequency of motion for Cases 1 and 3 is given by equation (73)

$$\frac{\omega_e^2}{\overline{M}_o} = \frac{\pi^2}{4} \frac{\frac{M_1 \sigma_m^2}{\overline{M}_o}\left(2 + \frac{\sigma_o^2}{\sigma_m^2}\right) + 1}{K(k_1)} \tag{A1}$$

The complete elliptic integral of the first kind may be expanded in the following series:

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdot \cdot \cdot\right] \tag{A2}$$

and for $k^2 < 1/2$, a sufficient approximation may be made by retaining only the first two terms. Thus

$$K(k) \simeq \frac{\pi}{2}\left(1 + \frac{1}{4} k^2\right) \tag{A3}$$

The modulus k_1^2 may be expressed as

$$k_1^2 = \frac{1 - \frac{\sigma_o^2}{\sigma_m^2}}{2 + \frac{\sigma_o^2}{\sigma_m^2} + \frac{\overline{M}_o}{M_1 \sigma_m^2}}$$

so that equation (3) may be written

$$K(k_1) = \frac{\pi}{2}\frac{1}{4}\left(4 + \frac{1 - \frac{\sigma_o^2}{\sigma_m^2}}{2 + \frac{\sigma_o^2}{\sigma_m^2} + \frac{\overline{M}_o}{M_1 \sigma_m^2}}\right) = \frac{\pi}{2}\frac{1}{4}\left(\frac{9 + 3\frac{\sigma_o^2}{\sigma_m^2} + 4\frac{\overline{M}_o}{M_1 \sigma_m^2}}{2 + \frac{\sigma_o^2}{\sigma_m^2} + \frac{\overline{M}_o}{M_1 \sigma_m^2}}\right) \tag{A4}$$

Substituting expression (4) into equation (A1), the new expression becomes

$$\frac{\omega_e^2}{\overline{M}_o} = 16\,\frac{\left[\frac{M_1\sigma_m^2}{\overline{M}_o} + \left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)\frac{M_1\sigma_m^2}{\overline{M}_o} + 1\right]^3}{\left[6\,\frac{M_1\sigma_m^2}{\overline{M}_o} + 3\left(\frac{\sigma_o^2}{\sigma_m^2}\right)\frac{M_1\sigma_m^2}{\overline{M}_o} + 4\right]^2} \tag{A5}$$

Dividing denominator into numerator, equation (45) becomes

$$\frac{\omega_e^2}{M_o} = 1 + \frac{3}{2}\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)\frac{M_1\sigma_m^2}{\overline{M}_o} + \frac{3}{16}\left(1 - \frac{\sigma_o^2}{\sigma_m^2}\right)^2\left(\frac{M_1\sigma_m^2}{\overline{M}_o}\right)^2 + \ldots \tag{A6}$$

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It can be seen from figure 3 that expression (A6) is a good approximation for Case 1 when only the first two terms are used. The first two terms are also a good approximation for Case 3 when $\sigma_o^2/\sigma_m^2 > 1/2$ as is seen in figure 5. For values of σ_o^2/σ_m^2 near zero, however, the linear segment of the curve approaches the value

$$\omega_e^2 = 1.107\overline{M}_o + 1.435\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)M_1\sigma_m^2 \tag{A7}$$

Expression (A7) was determined by measuring the slope of the linear segment of the $\sigma_o^2/\sigma_m^2 = 0$ curve in figure 5.

The series expansion for Case 2 may be found in a manner analogous to that of Cases 1 and 3. Equation (A3) will again be used as an approximation to the complete elliptic integral of the first kind. For Case 2, however, the modulus k_2^2 is expressed as

$$k_2^2 = -\,\frac{\frac{M_1\sigma_m^2}{\overline{M}_o}\left(1 - \frac{\sigma_o^2}{\sigma_m^2}\right)}{\frac{M_1\sigma_m^2}{\overline{M}_o}\left(1 + 2\,\frac{\sigma_o^2}{\sigma_m^2}\right) + 1}$$

Substituting this value into equation (A3), the expression may be written

$$K(k_2) = \frac{\pi}{2}\,\frac{1}{4}\left[\frac{3\,\frac{M_1\sigma_m^2}{\overline{M}_o} + 9\left(\frac{M_1\sigma_m^2}{\overline{M}_o}\right)\frac{\sigma_o^2}{\sigma_m^2} + 4}{\frac{M_1\sigma_m^2}{\overline{M}_o}\left(1 + 2\,\frac{\sigma_o^2}{\sigma_m^2}\right) + 1}\right]$$

Substituting this value into equation (79) the equation for the frequency becomes

$$\frac{\omega_e^2}{\overline{M}_o} = 16\,\frac{\left[\frac{M_1\sigma_m^2}{\overline{M}_o}\left(1 + \frac{2\sigma_o^2}{\sigma_m^2}\right) + 1\right]^3}{\left[3\,\frac{M_1\sigma_m^2}{\overline{M}_o}\left(1 + 3\,\frac{\sigma_o^2}{\sigma_m^2}\right) + 4\right]^2} \tag{A8}$$

and if the denominator is divided into the numerator by term, equation (A8) may be written in series form as

$$\frac{\omega_e^2}{\overline{M}_o} = 1 + \frac{3}{2}\left(1 + \frac{\sigma_o^2}{\sigma_m^2}\right)\frac{M_1\sigma_m^2}{\overline{M}_o} + \frac{3}{16}\left(1 - \frac{\sigma_o^2}{\sigma_m^2}\right)^2\left(\frac{M_1\sigma_m^2}{\overline{M}_o}\right)^2 + \ldots \tag{A9}$$

The first three terms of this series are exactly the same as were obtained for the first three terms of the series for Cases 1 and 3. It may also be seen that only the first two terms of equation (A9) describe the linear segments of the curves in figure 4.

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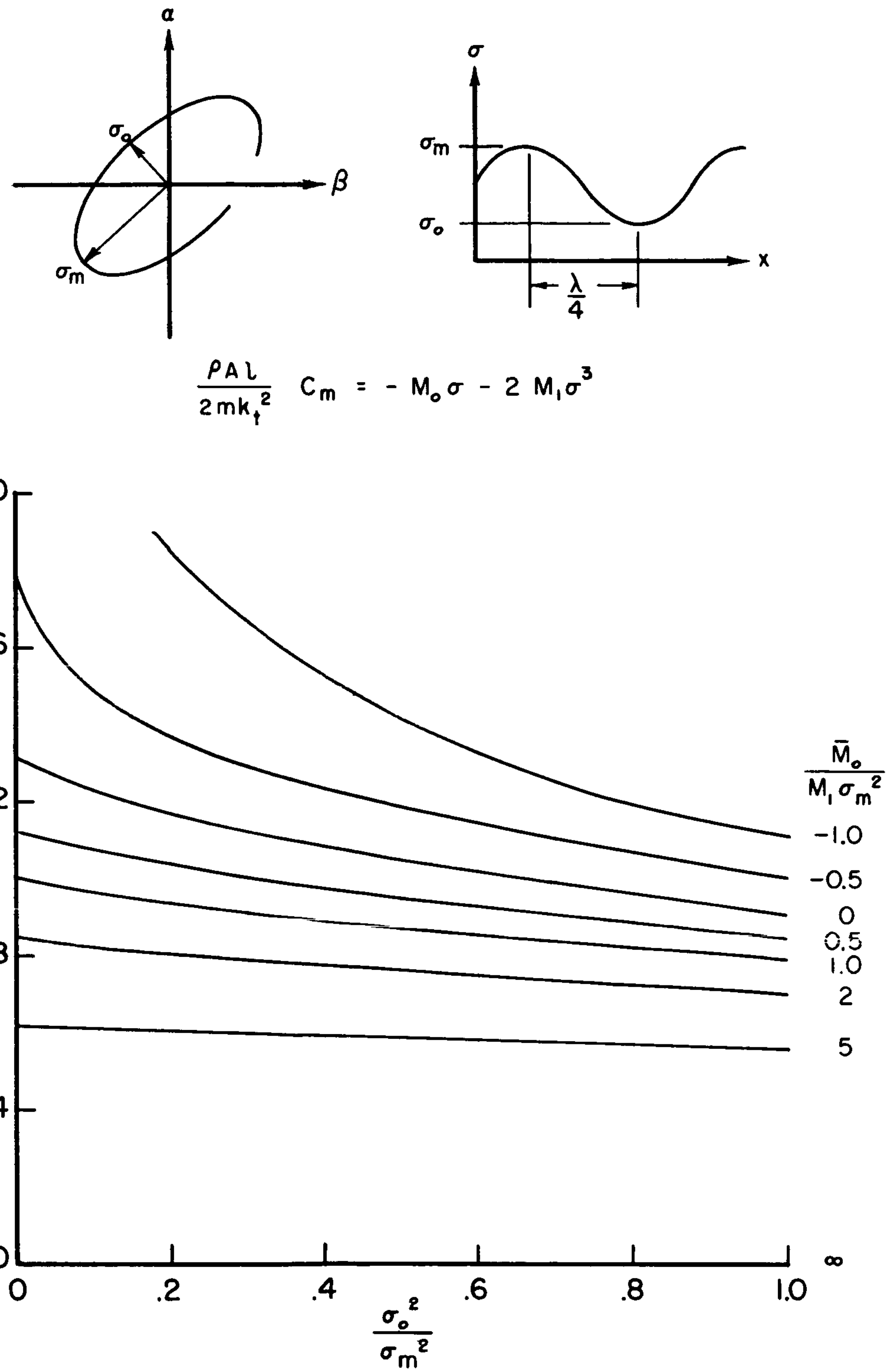

Figure 1.- Wave-length parameter as a function of amplitude ratio for Cases 1 and 3.

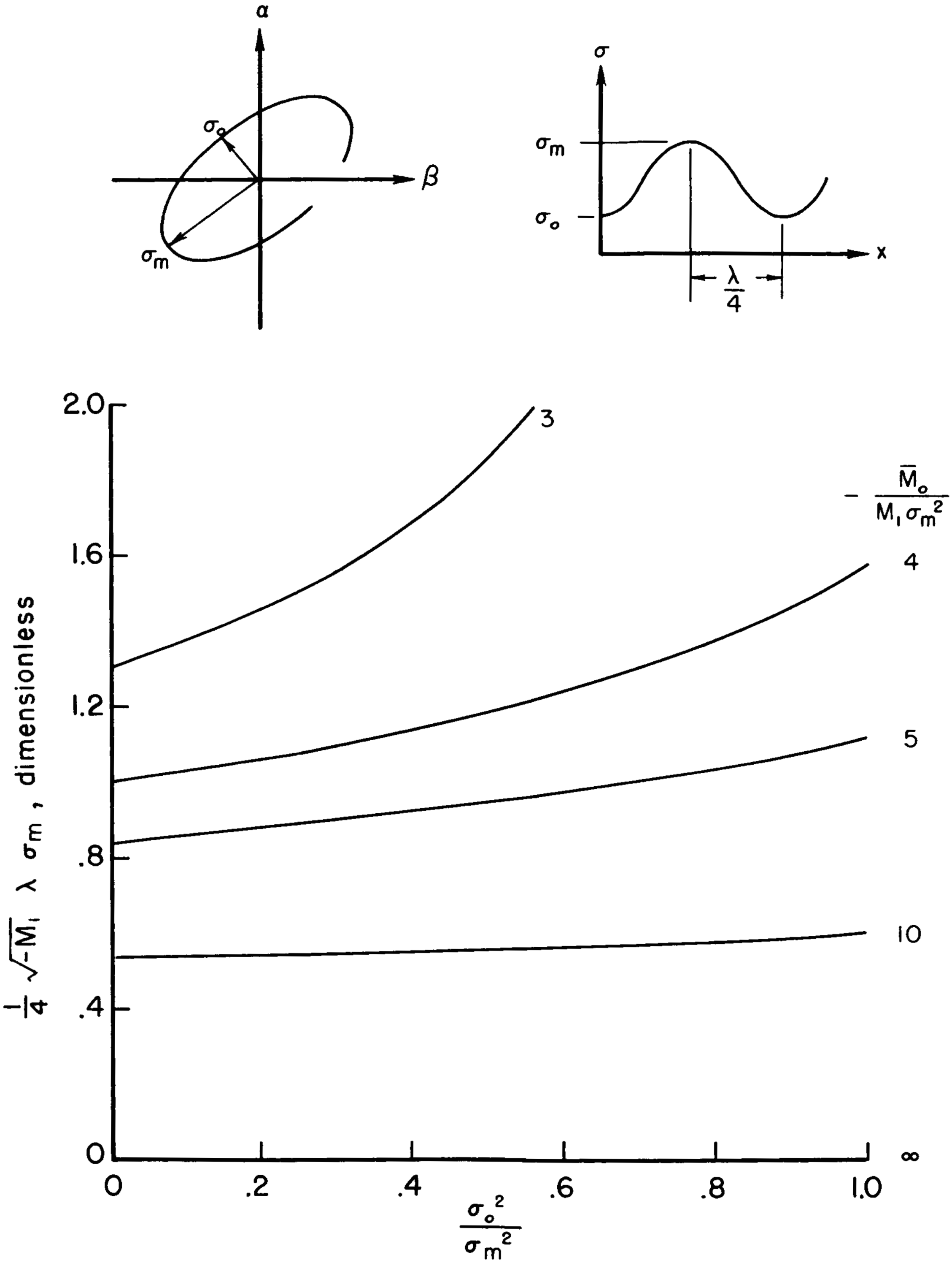

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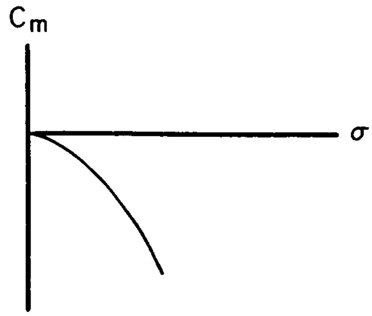
Figure 2.- Wave-length parameter as a function of amplitude ratio for Case 2.



$$\frac{\rho A l}{2mk_t^2} \; C_m = -M_o \sigma - 2 M_1 \sigma^3$$

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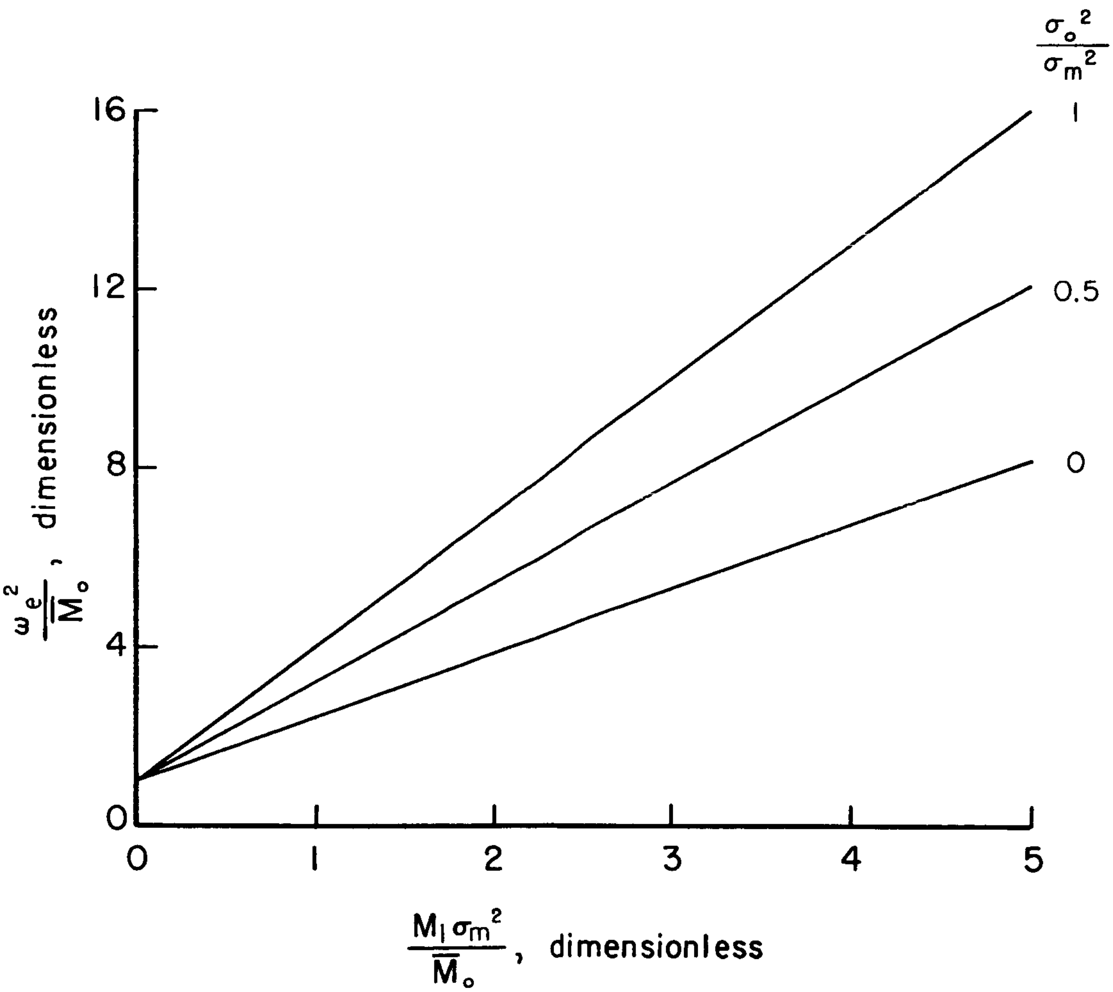

Figure 3.- Frequency parameter versus amplitude parameter for Case 1.

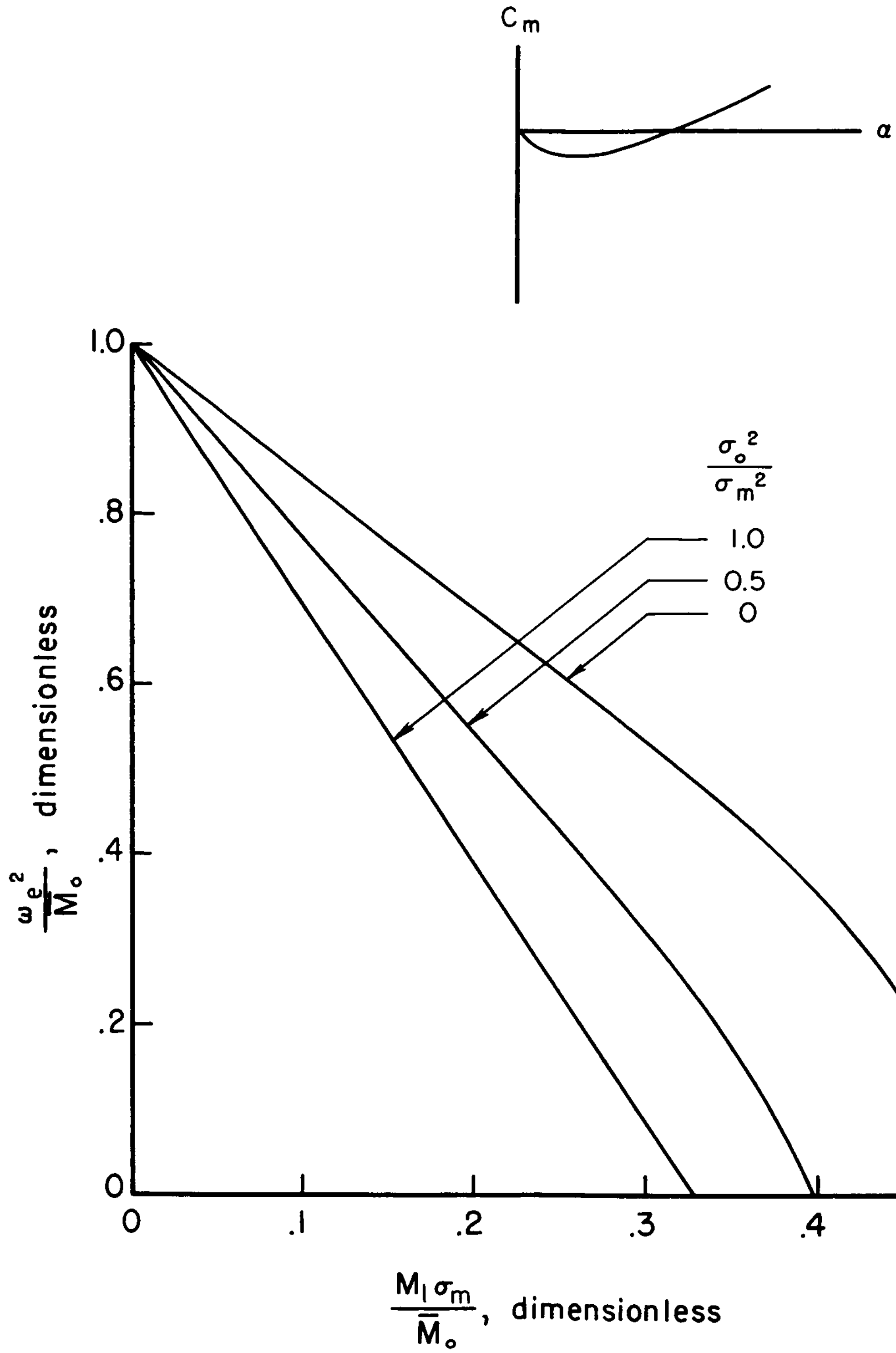

Figure 4.- Frequency parameter versus amplitude parameter for Case 2.

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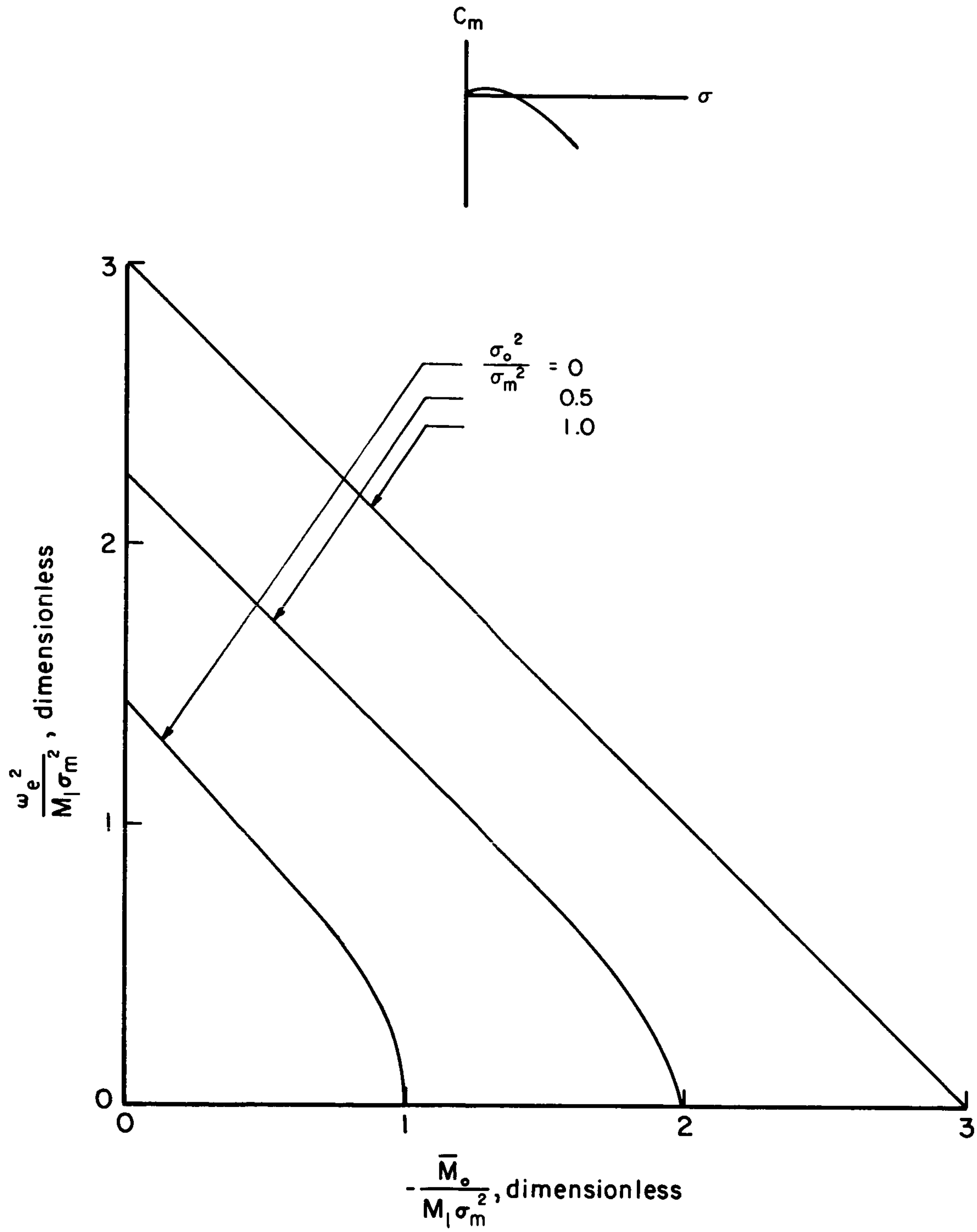

Figure 5.- Frequency parameter versus amplitude parameter for Case 3.

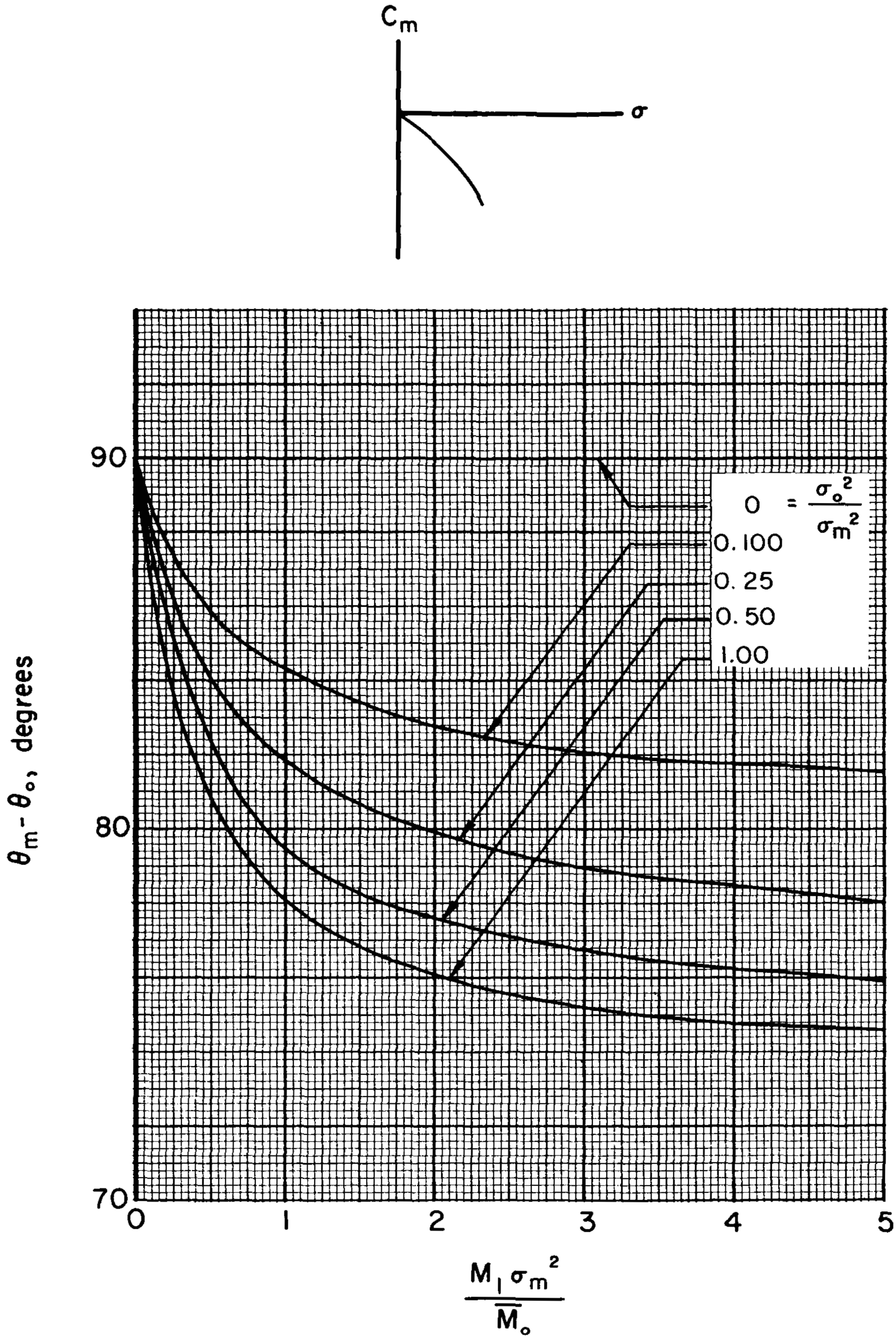

Figure 6.- Polar rotation for a quarter period for Case 1.

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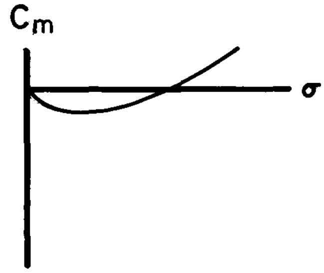

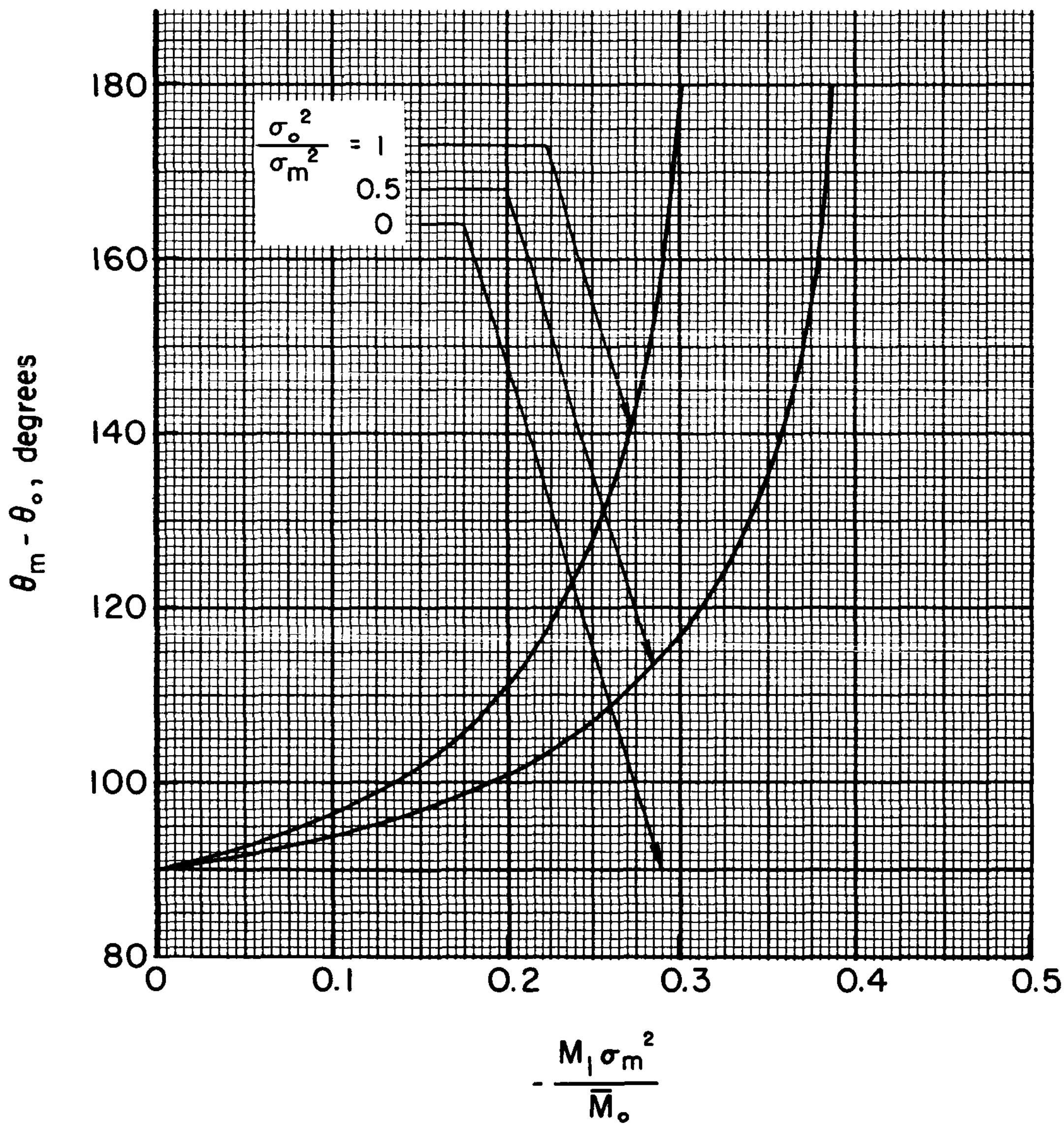

Figure 7.- Polar rotation for a quarter period for Case 2.

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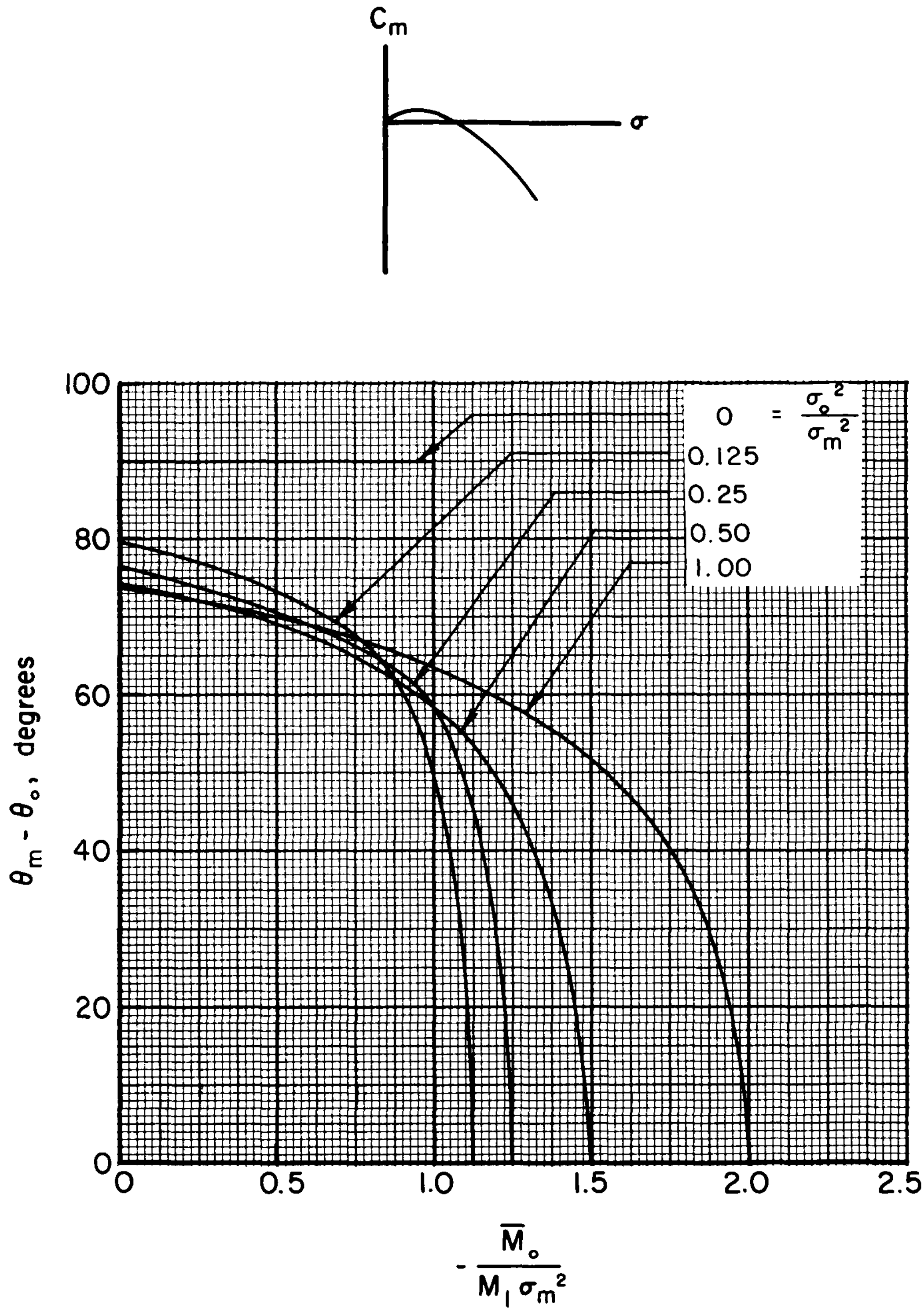

Figure 8.- Polar rotation for a quarter period for Case 3.

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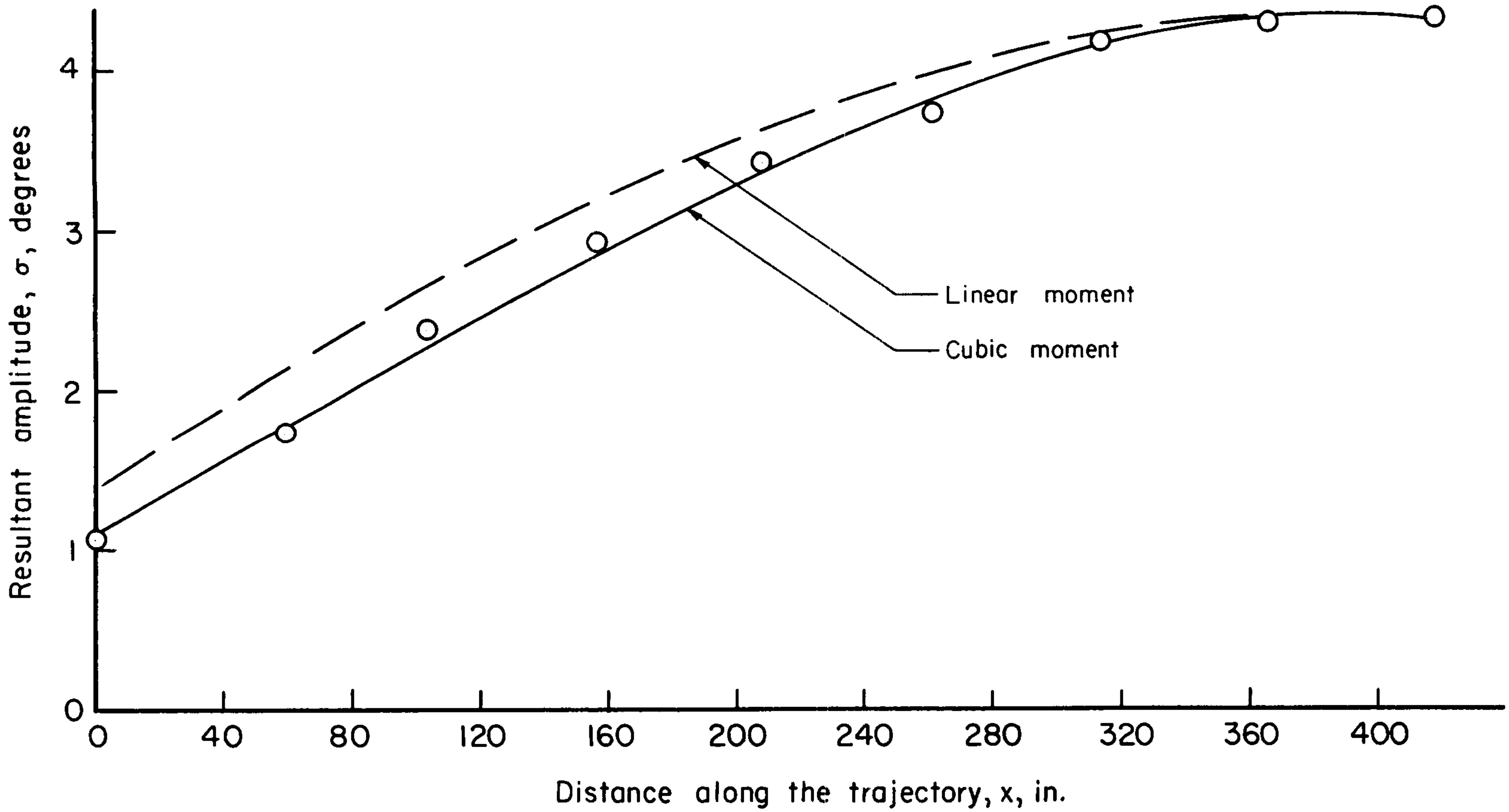

Figure 9.- Resultant amplitude for run B.

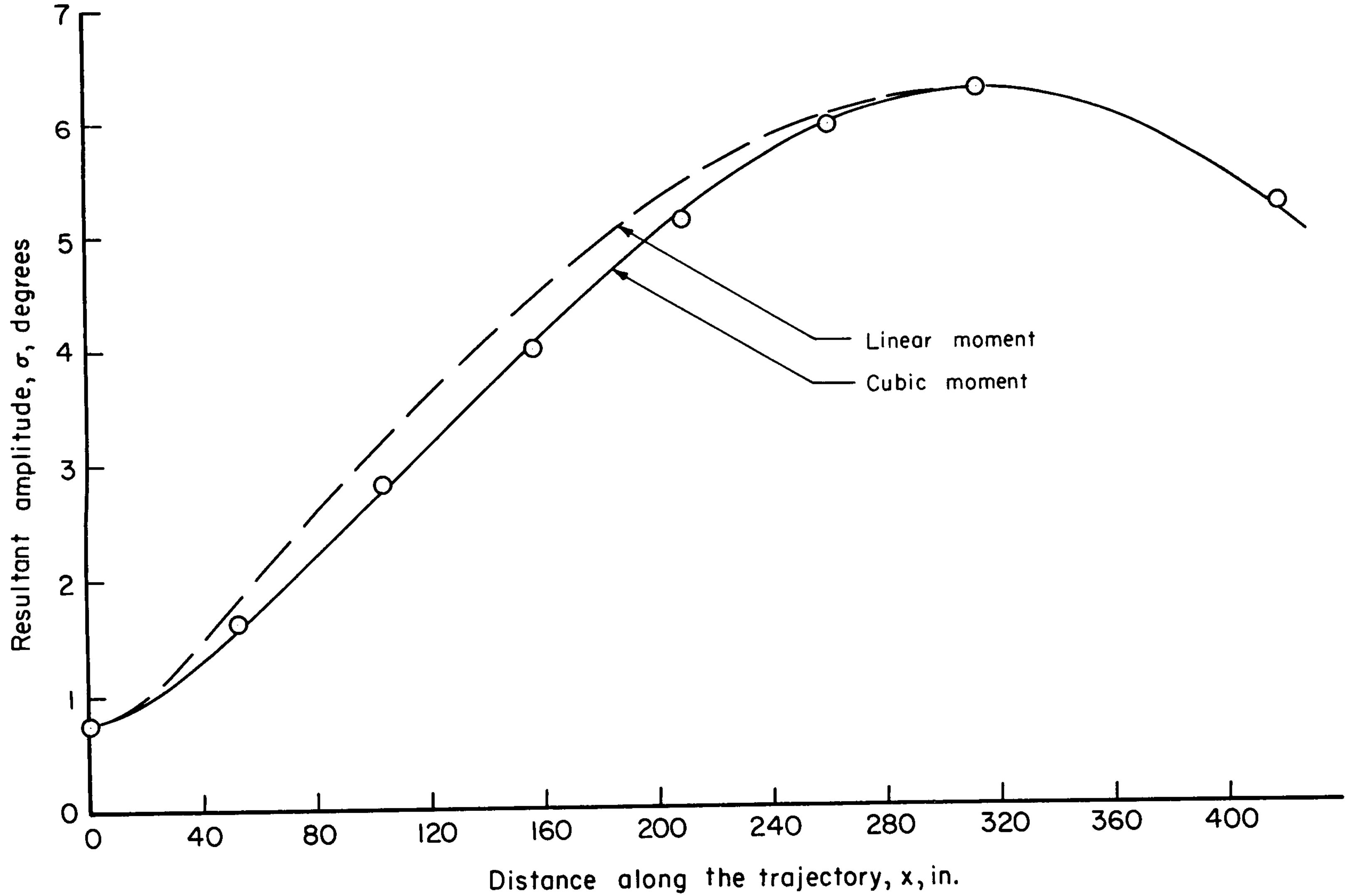

Figure 10.- Resultant amplitude for run A.

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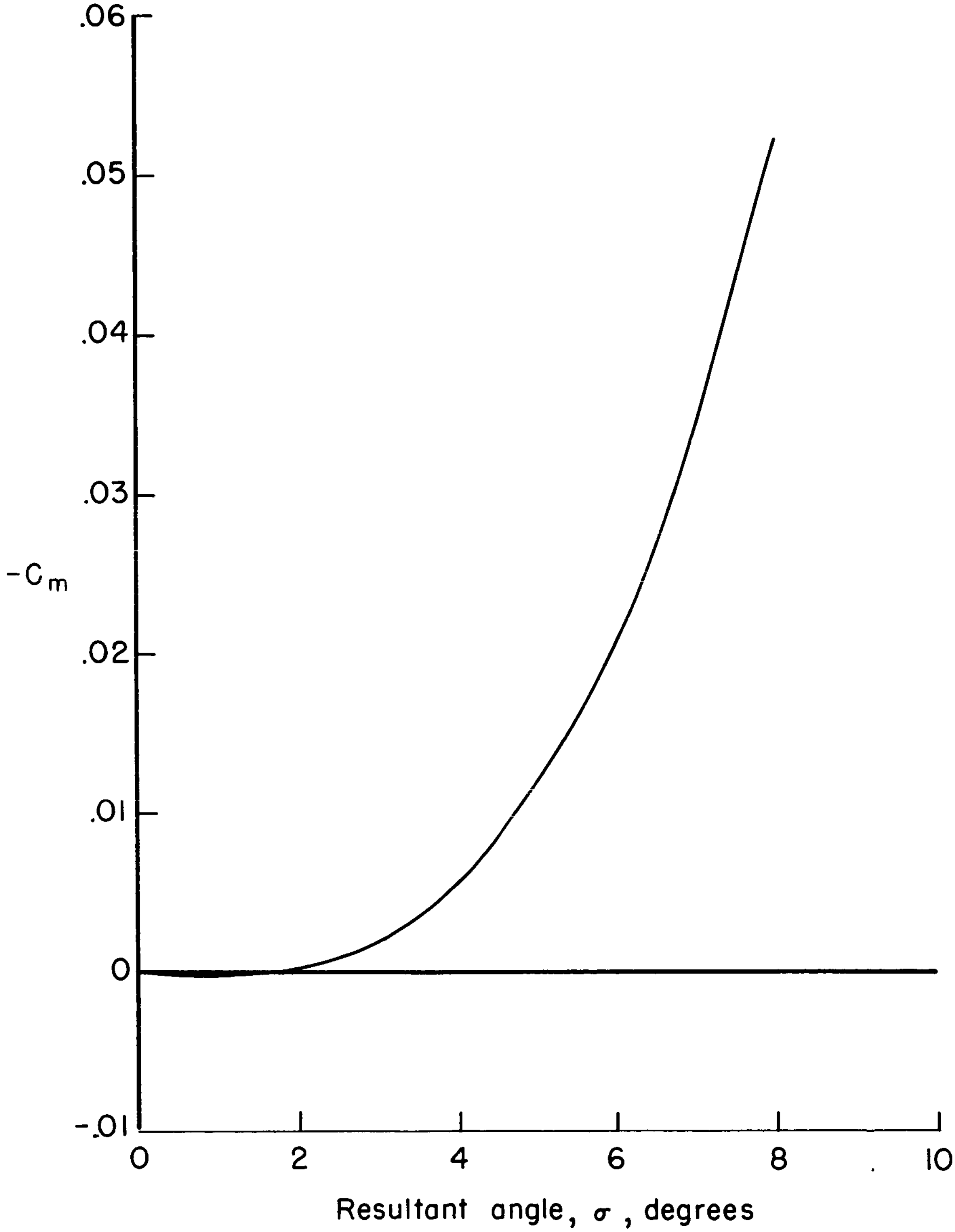

Figure 11.- Moment curve for runs A and B.